AM 258

Computational Fluid Dynamics

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VORTICITY PRODUCTION AT A SOLID BOUNDARY

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Is vorticity flux a quantity that can be unambiguously defined?

If not, does it matter which definition is used?

Or, more generally: What is the best way to describe vorticity production at a boundary?

It is noted here that the definition of vorticity flux by analogy with heat flux is not the only possible definition. An alternate definition is proposed which has some advantages.

Vorticity production at a solid boundary can be described in terms of the vorticity flux, which was defined by Lighthill [1] as

$$-\nu\frac{\partial\vec{\omega}}{\partial z} \qquad (1)$$

for the plane surface $z = 0$. Note that the analogy to Fourier's law is strictly valid only in two-dimensional flows. Panton [2] generalized this to an arbitrary surface in three dimensions by defining it as

$$-\nu(\vec{n}\cdot\nabla)\vec{\omega} \qquad (2)$$

where n is the unit normal vector to the surface, directed into the fluid. This definition is consistent with the result of integrating the viscous diffusion term in the vorticity equation over a control volume, because

$$\int \nu\nabla^2\vec{\omega}\,dV = -\int \nu(\vec{n}\cdot\nabla)\vec{\omega}\,dS \qquad (3)$$

and the integrand on the right side can be identified as the vorticity flux.

The definition of vorticity flux by analogy with heat flux is not the only possible definition, however. An alternate expression for the vorticity production at a boundary also follows from integrating the viscous diffusion term over a control volume, using the identity

$$-\int \nu\nabla\times(\nabla\times\vec{\omega})\,dV = \int \nu\vec{n}\times(\nabla\times\vec{\omega})\,dS \qquad (4)$$

Thus one may define the rate of vorticity production per unit area of the surface (the term vorticity flux is not appropriate for this quantity, because it is tangent to the boundary) as [3]

$$\nu\vec{n}\times(\nabla\times\vec{\omega}) \qquad (5)$$

Equations (2) and (5) are two different definitions of the rate of vorticity production per unit area of a surface. The fact that they are not equivalent is easily seen by examining their components at a plane surface. When integrated over a closed control surface, however, both definitions give the same result, because it can be shown [4] that

$$\int(\vec{n}\cdot\nabla)\vec{\omega}\,dS = -\int\vec{n}\times(\nabla\times\vec{\omega})\,dS \qquad (6)$$

The definition (5) has certain advantages. First, since $-\nu\nabla\times\vec{\omega}$ is the net viscous force per unit mass of a fluid element, (5) implies that the vorticity production is proportional to the component of this force tangent to the surface. Second, by using the Navier-Stokes equation it is possible to write it as follows in terms of the velocity of a point on the surface and the pressure gradient at the surface:

$$\nu\vec{n}\times(\nabla\times\vec{\omega})_0 = -\vec{n}\times\left[(1/\rho)(\nabla p)_0 + d\vec{V}/dt\right] \qquad (7)$$

The result was given for two-dimensional flows by Morton [5], who adopted Lighthill's definition of the vorticity flux. The definition (5) enables one to obtain the same result in three dimensions, but (2) does not. The physical interpretation of Eq. (7) is that the acceleration of the surface and the pressure gradient in a plane tangent to the surface, rather than viscosity, are responsible for vorticity production. Viscosity plays only an indirect role in enforcing the no-slip condition.

Does it matter that there is more than one way to define the vorticity production at a solid surface? This concept explains where the vorticity comes from, but Lighthill [1] thought that vorticity flux was valueless as a boundary condition, because vorticity production at the boundary determined the normal gradient and not vice versa. Recently there has been renewed interest in numerical methods using vorticity creation boundary conditions (see [6] and the references cited therein). It is too early to tell whether such methods offer significant advantages over existing methods. If they do, then there may be interest in examining whether the alternative definition of vorticity production at a solid boundary proposed here makes any differences, especially when the method is extended to three-dimensional flows. The direct relation (7) between vorticity production and the surface pressure gradient and acceleration in three-dimensional flows might favor the alternate definition.

A more general question concerns the dynamics of vorticity production. Vorticity is a kinematical quantity, and the equation governing its evolution is derived from the Navier-Stokes equation by a purely mathematical operation, so that the physical meaning of the various terms is not immediately clear. Since the analogy with the heat equation is not exact, terms such as the vortex tilting term and the vorticity production term are difficult to explain. In view of this the ambiguity in the definition of vorticity production is not too surprising.

1. Lighthill, M.J., Chapter II of Laminar Boundary Layers (ed. L. Rosenhead), Oxford University Press, 1963, pp. 54-60.
2. Panton, R. L., Incompressible Flow, Wiley, 1984, Secs. 13.7, 13.11.
3. Lyman, F. A., Unpublished course notes, Syracuse University, 1972
4. Panton, R. L., Private communication, Nov. 18, 1985.
5. Morton, B. R., "The Generation and Decay of Vorticity," Geophys. Astrophys. Fluid Dynamics 28, 1984, pp. 277-308.
6. Anderson, C.R., "Vorticity Boundary Conditions and Boundary Vorticity Generation for Two-Dimensional Viscous Incompressible Flows," J. Comp. Phys. 80, 1989, pp. 72-97.

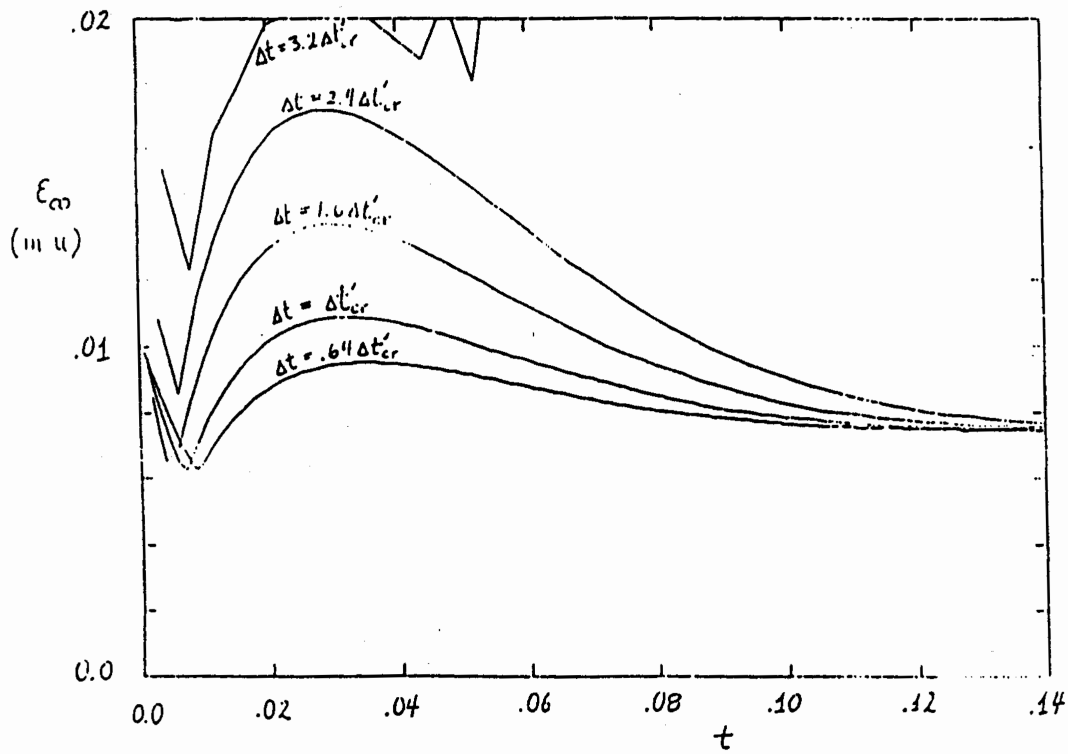


Figure 2: Model problem, semi-implicit method $\left(\Delta y = .05, \Delta t'_{cr} = \frac{\Delta y^2}{2} = .00125\right)$

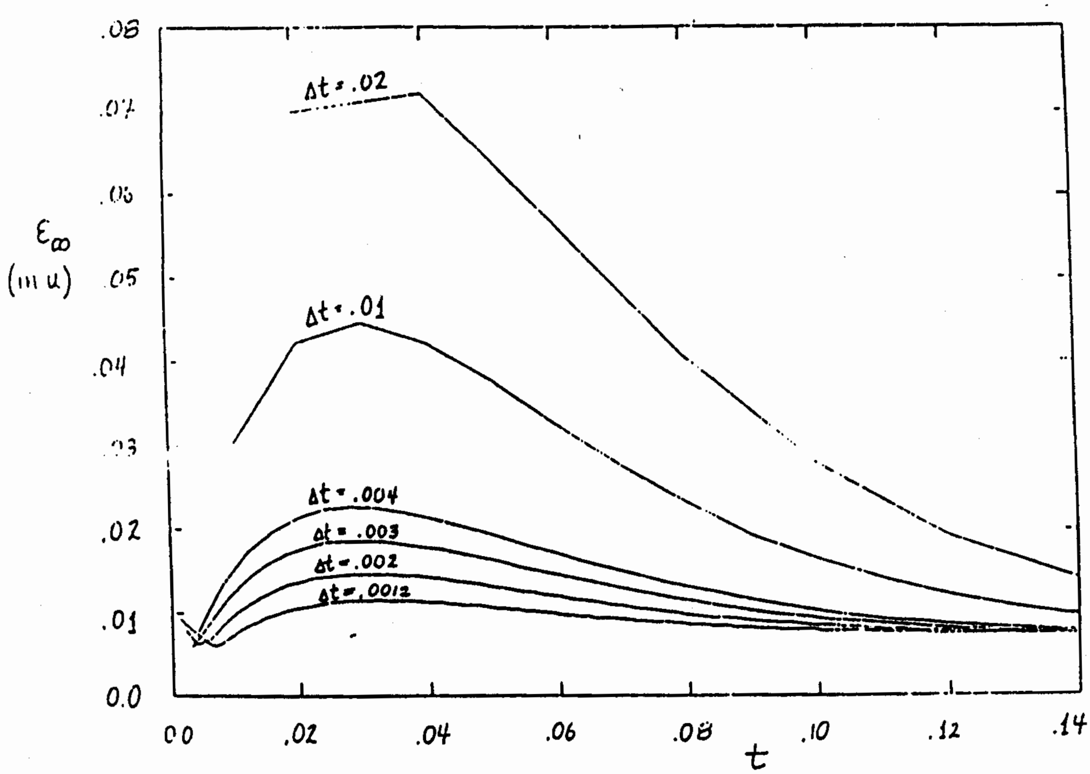


Figure 3: Model problem, full implicit, Green's function $(\Delta y = .05)$

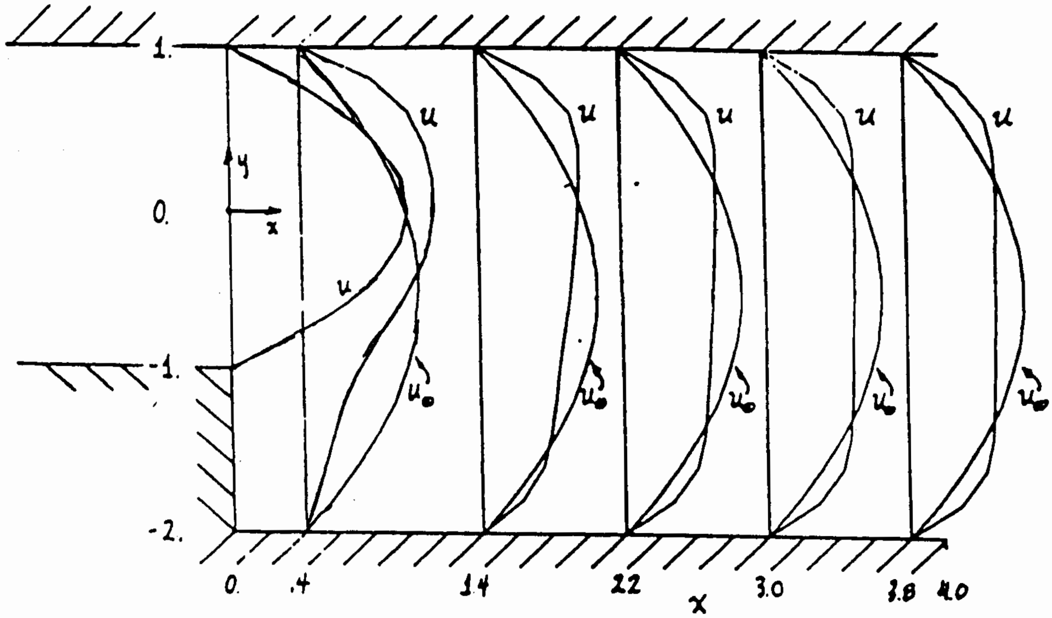


Figure 1a: Stokes Flow-Streamwise Velocity (t=.02).

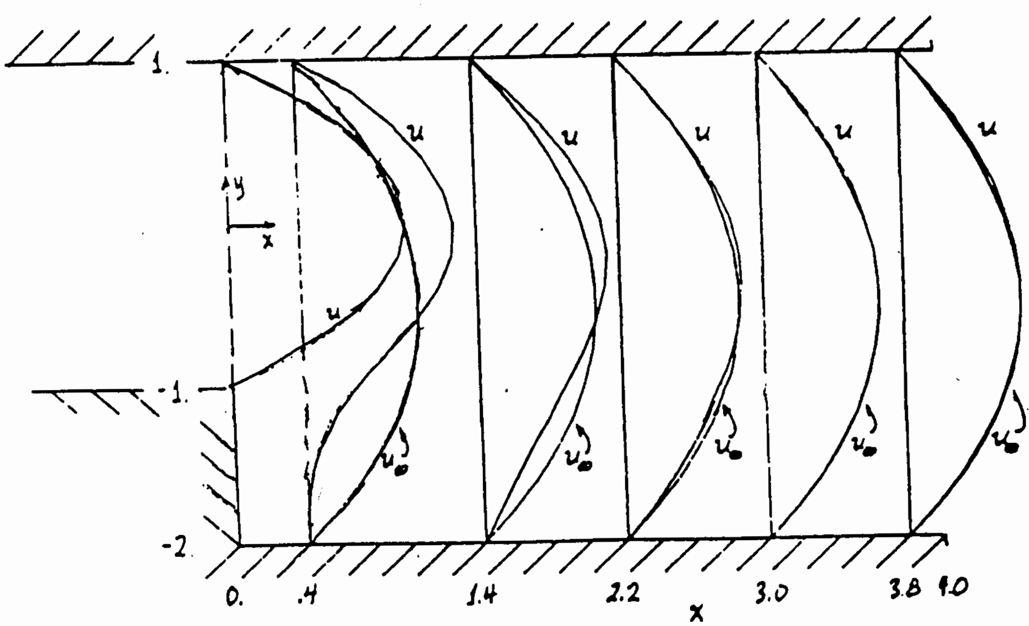


Figure 1b: Stokes Flow-Streamwise Velocity (t=1.4)

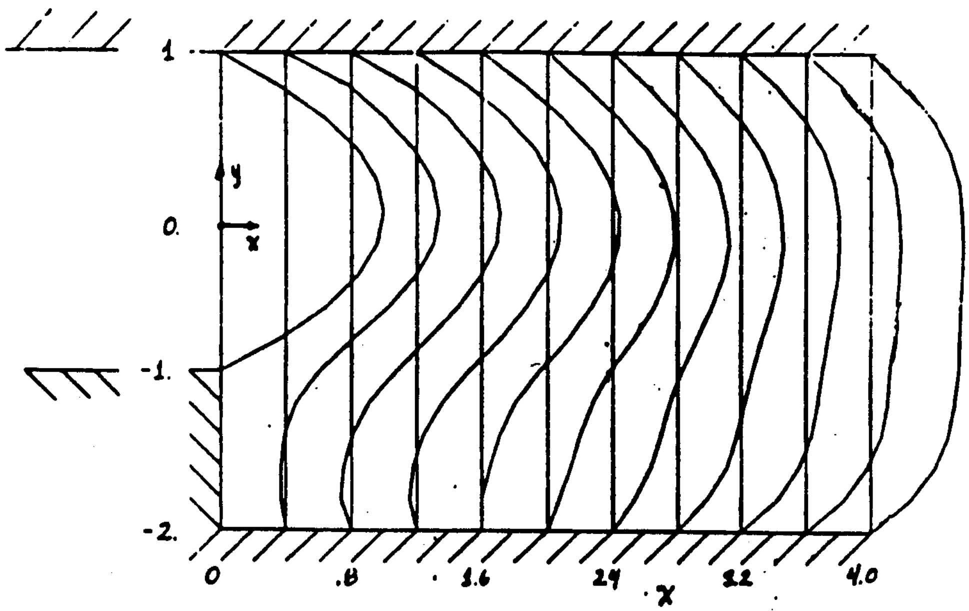


Figure 2a: Navier-Stokes — Streamwise Velocity (t=5.33)

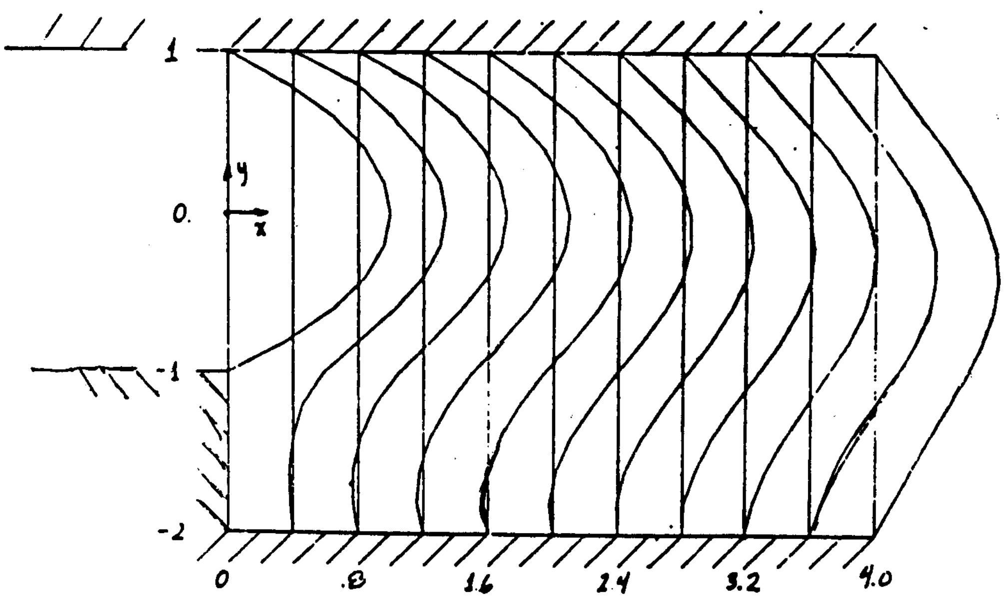


Figure 2a: Navier-Stokes — Streamwise Velocity (t=13.33)

$$+B_{ijk}\frac{d\overline{\Psi}_j}{dy}\frac{d^2\overline{\Psi}_k}{dy^2} - C_{ijk}\overline{\Psi}_j\frac{d\overline{\Psi}_k}{dy} - B_{ijk}\frac{d^3\overline{\Psi}_k}{dy^3}\overline{\Psi}_j\Big)\Big]$$

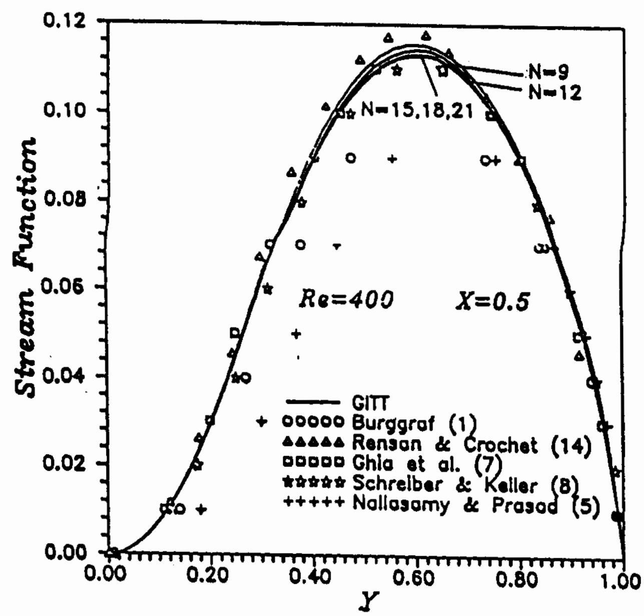
The boundary conditions for the transformed variable are:

$$\overline{\Psi}_i(0) = 0, \quad \frac{d\overline{\Psi}_i}{dy}(0) = 0$$

$$\overline{\Psi}_i(1) = 0, \quad \frac{d\overline{\Psi}_i(1)}{dy} = \int_o^1 \phi_i(x)dx = \begin{cases} \dfrac{4}{\mu_i}\tan\dfrac{\mu_i}{2}, i \text{ odd} \\ 0, i \text{ even} \end{cases}$$

This is a system of an infinite set of nonlinear equations (4^{th} order ODEs) which can be truncated to order N and solved using standard ODE packages. Convergence is obtained as N increases.
The plot shows the streamfunctin profile for different N at the midsection of the cavity. For $N \geq 2$ the results have converged.

Reference: J.S. Perez Guerrero and R.M. Cotta, "Integral transform solution for the lid-driven cavity flow problem in streamfunction-only formulation," Int. J. Num. Meth. Fluids, vol. 15, p. 399–409, 1992.

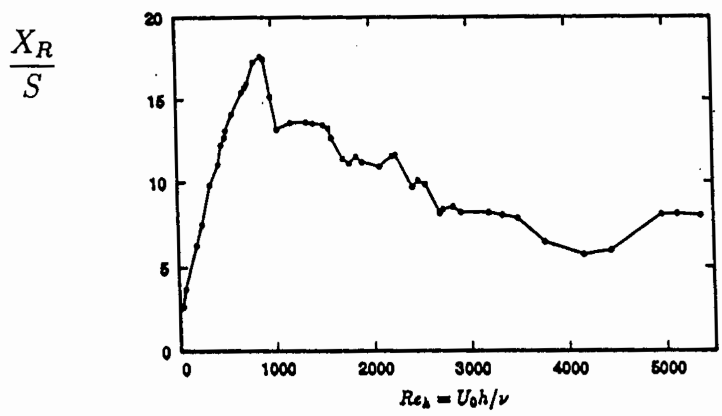


which *does not give any separation*!! If the same condition is imposed at station (2) $\Rightarrow$ recover separation. For $Re < 300$ upstream location important (Kaiktsis, 1991); $Re > 300$ it does not matter.

Reference: Armaly, B.F., Durst, F. Pereira, J.C.F. and Schönung, B., 1983, "Experiment and theoretical investigation of backward-facing step," J. of Fluid Mech., vol. 127, pp. 473–496.

Step Flow: Benchmark Problem:

The flow over a backward-facing step has been used as a validation problem in many CFD studies including the laminar regime, transition, and fully turbulent region where new turbulence models are typically tested. The variation of the reattachment length (size of primary recirculation zone) is shown in the Figure as a function of Reynolds number (defined with the maximum inlet velocity U_0 and the step height h). This plot is constructed based on the experimental measurements of Armaly et al. (1983).



For $Re_h < 900$ the flow is laminar, for $900 \leq Re_h \leq 5000$ the flow is transitional, and for $Re_h > 5,000$ the flow is fully turbulent. CFD codes are tested usually for $Re_h < 500$ where the flow is steady and laminar while turbulence models are tested for $Re_h > 5,000$ where there is no longer effect of Re_h on the reattachment length.