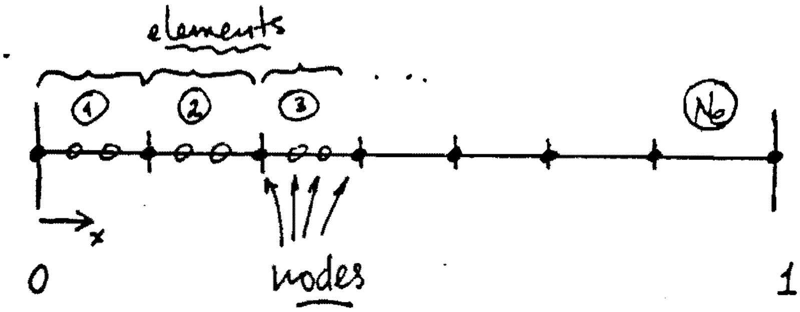
(1)

G. Finite - Element Method (Local WRT)(Variational)

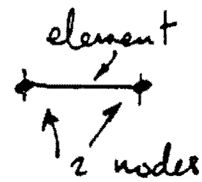
$$\left(\frac{d^2\theta}{dx^2} = q \qquad \theta(0) = \theta(1) = 0 \right)$$

1. Domain discretization



In the finite-element discretization, we break the domain up into elements (1, 2, 3, ..., N_e above). Associated with each element are nodes.

linear finite element basis :



⇒ each node belongs to two elements.

(higher order basis (1-D) ⇒ internal nodes o in an element).

2. Local - Co-ordinate System

In what follows, superscript (f^k) refers to f in element (k).

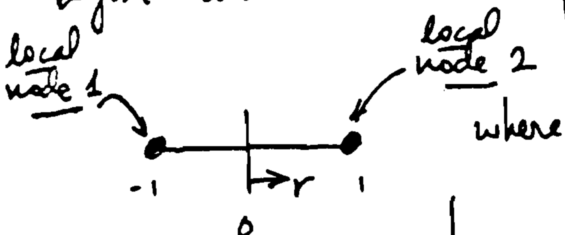
In 1-D define position of element k as x_L^k, x_R^k i.e.

$\underline{x_L^k \le x \le x_R^k \text{ in element } k}$. Also $L^k = x_R^k - x_L^k$

$x = \frac{L^k \cdot r}{2} + x_L^k + \frac{L^k}{2}$

$x = \frac{L^k}{2}(r+1) + x_L^k$

Define local co-ordinate system



where $-1 \le r \le 1$, so

$$\left\{ r = \underbrace{\frac{2(x - x_L^k)}{L^k}} - 1 \right\} \left\{ \begin{array}{l} x_L^k \rightarrow r = -1 \\ x_R^k \rightarrow r = +1 \end{array} \right\}$$

note $\frac{d}{dx} = \frac{2}{L^k}\frac{d}{dr}$, $dx = \frac{L^k}{2} dr$. $\frac{dr}{dx} = \frac{2}{L^k}$

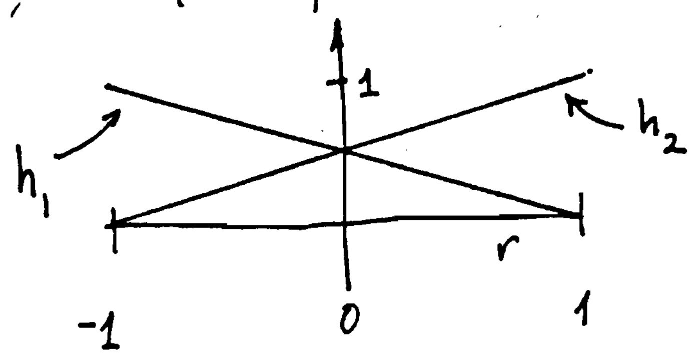
3. Local Interpolation (Basis) Functions (WRT)

(linear finite element basis) (subscripts refer to nodes)

In element k define

$\theta^k = \sum_{i=1}^{2} \theta_i^k h_i(r)$ h_i interpolation functions

expansion coeff.



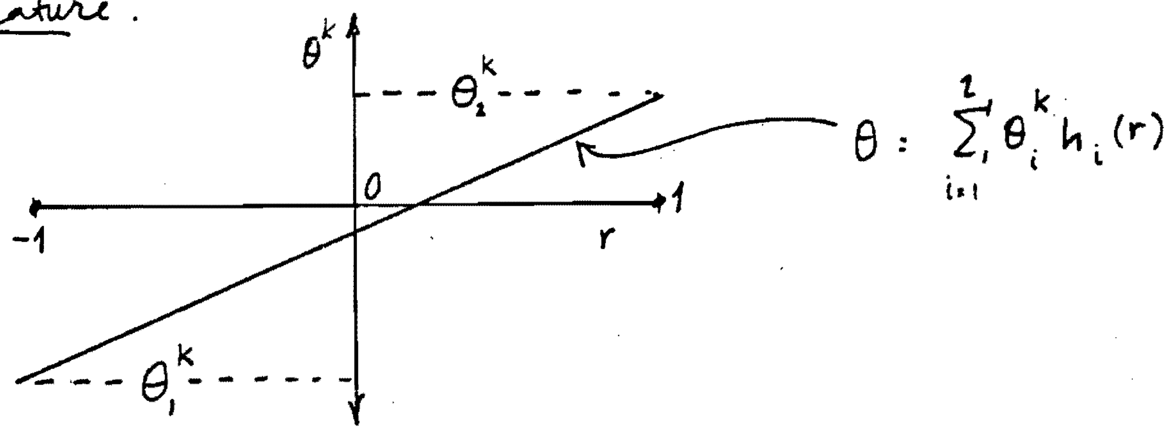
③

So the h_i are Lagrangian interpolants $= \begin{cases} 1 & \text{node } i \text{ (local)} \\ 0 & \text{node } j \neq i \end{cases}$

$$h_1 = -\frac{1}{2}(-1+r) = \frac{1}{2}(1-r)$$

$$h_2 = \frac{1}{2}(1+r) = \frac{1}{2}(1+r)$$

{ The θ_i^k are the <u>basis function</u> coefficients, but also, because of the way the h_k are defined, θ_i^k is the value of the (numerical) solution at the i^{th} <u>local</u> node, i.e. the <u>nodal temperature</u>. }



4. Elemental Equations

For the moment we consider just the element ⓚ. To formulate a Rayleigh-Ritz procedure in element ⓚ the functional is therefore

Galerkin projection or Variationally

④

$$\Pi^k(\theta^k) = \int_{x_L^k}^{x_R^k} \left\{ -\frac{1}{2}\left(\frac{d\theta}{dx}\right)^2 - \theta q \right\} dx$$

which, in the local coordinate system, becomes

$$\Pi^k(\theta^k) = \int_{-1}^{1} \left\{ -\frac{1}{2}\left(\frac{2}{L^k}\right)^2 \left(\frac{d\theta}{dr}\right)^2 - \theta q \right\} \left(\frac{L^k}{2}\right) dr .$$

Then

$$\delta_\theta \Pi^k = \int_{-1}^{1} \left\{ -\left(\frac{2}{L^k}\right)^2 \frac{d\delta\theta}{dr}\frac{d\theta}{dr} - q\,\delta\theta \right\} \left(\frac{L^k}{2}\right) dr$$

For the Rayleigh-Ritz method (linear basis)

Like test function in Galerkin projection

$$\theta^k(r) = \sum_{i=1}^{2} \theta_i^k h_i(r) \quad , \quad \delta\theta^k(r) = \sum_{i=1}^{2} \delta\theta_i^k h_i(r)$$

and

$$\delta_\theta \Pi^k = \int_{-1}^{1} \left\{ -\left(\frac{2}{L^k}\right)^2 \sum_{i=1}^{2} \delta\theta_i^k \frac{dh_i}{dr} \sum_{j=1}^{2} \theta_j^k \frac{dh_j}{dr} - q \sum_{i=1}^{2} \delta\theta_i^k h_i \right\} \left(\frac{L^k}{2}\right) dr$$

$$= \sum_{i=1}^{2} \delta\theta_i^k \left(\sum_{j=1}^{2} A_{ij}^k \theta_j^k - Q_i^k \right)$$

$A_{i\cdot}^k$: equation corresponding to variation in θ_i^k

where

$$A_{ij}^{k} = \frac{-2}{L^{k}} \int_{-1}^{1} \frac{dh_i}{dr}\frac{dh_j}{dr}\,dr \quad , \quad Q_i^{k} = \frac{L^{k}}{2}\int_{-1}^{1} q\, h_i \, dr$$

note A_{ij}^{k} symmetric

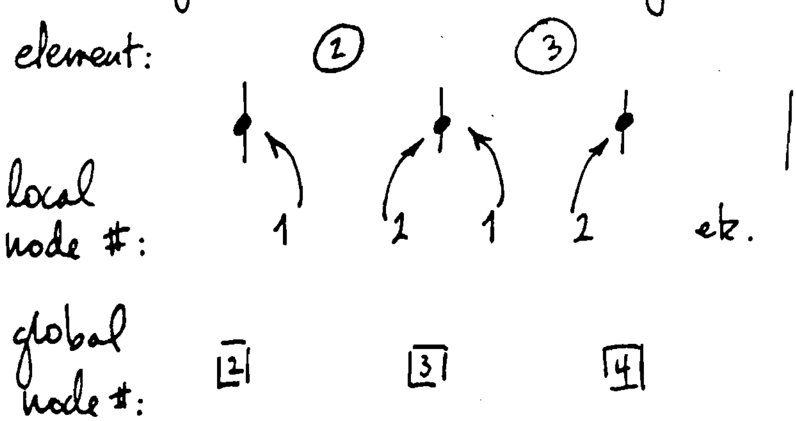
We can evaluate A_{ij}^{k} ; $\frac{dh_1}{dr} = -\frac{1}{2}$, $\frac{dh_2}{dr} = \frac{1}{2}$

$$\Rightarrow \quad A_{ij}^{k} = \frac{1}{L^{k}} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

If we had just one element, upon requiring that $\delta\Pi^{k} = 0$ we would get the equations $A_{ij}^{k}\,\theta_j^{k} = q_i^{k}$ as before. How do we take into account contributions from all the matrices?

5. "Direct Stiffness" Method

* global node numberings



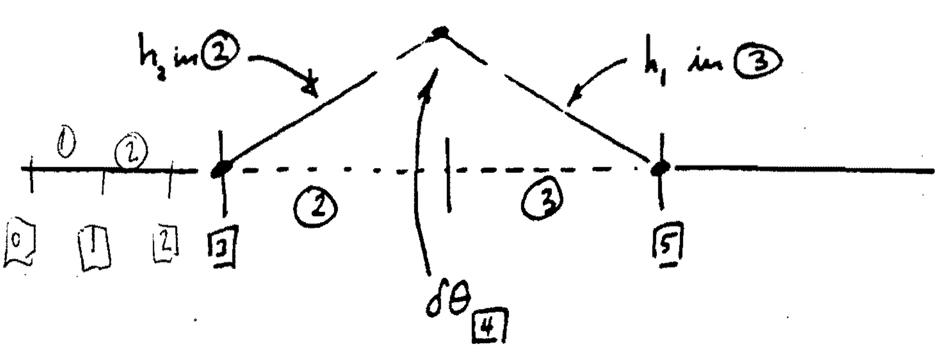
the global node numbers are required to connect the elements and satisfy continuity requirements

$$\theta_1^{③} = \theta_2^{②} = \theta_{\boxed{3}}$$ (only defined once – nodal temperature)

Unlike DGM

6

* basis functions re-visited



wrong sketch?

wrong (Rectly numbering)

In fact, a variation in $\theta_{\boxed{4}}$ corresponds to $\delta\theta_2^{(2)}$ and $\delta\theta_1^{(3)}$; given our representation of $\delta\theta^k = \sum_{i=1}^{2} \delta\theta_i^k h_i(r)$ this implies that our basis functions <u>globally</u> look like tophats that span two elements.

For the two elements ② and ③

$$\delta\Pi^{(2)} + \delta\Pi^{(3)} = \sum_{i=1}^{2} \delta\theta_i^{(2)} \left(\sum_{j=1}^{2} A_{ij}^{(2)} \theta_j^{(2)} - Q_i^{(2)} \right) + \sum_{i=1}^{2} \delta\theta_i^{(3)} \left(\sum_{j=1}^{2} A_{ij}^{(3)} \theta_j^{(3)} - Q_i^{(3)} \right)$$

$$= \delta\theta_{\boxed{3}} \left(\sum_{j=1}^{2} A_{1j}^{(2)} \theta_j^{(2)} - Q_1^{(2)} \right) \qquad \left(\delta\theta_{\boxed{3}} = \delta\theta_1^{(2)} \right)$$

$\boxed{2}$

$$+ \delta\theta_{\boxed{4}} \left(\sum_{j=1}^{2} A_{2j}^{(2)} \theta_j^{(2)} + \sum_{j=1}^{2} A_{1j}^{(3)} \theta_j^{(3)} - Q_2^{(2)} - Q_1^{(3)} \right)$$

$\boxed{3}$

$$+ \delta\theta_{\boxed{5}} \left(\sum_{j=1}^{2} A_{2j}^{(3)} \theta_j^{(3)} - Q_2^{(3)} \right)$$

$\boxed{4}$

(7)

Now the contributions for $\delta\Theta_{\boxed{3}}$, $\delta\Theta_{\boxed{5}}$ are not complete (they will be affected by elements (1) and (4), respectively). However, we have now taken into account "all" of $\delta\Theta_{\boxed{4}}$, and the equation reads (in global references)

$$\delta\Theta_{\boxed{4}}\left(A^{(2)}_{21}\,\Theta_{\boxed{3}} + A^{(2)}_{22}\,\Theta_{\boxed{4}} + A^{(3)}_{11}\,\Theta_{\boxed{4}} + A^{(3)}_{12}\,\Theta_{\boxed{5}} - Q^{(2)}_{2} - Q^{(3)}_{1}\right) \longrightarrow \text{Eq 4}$$

So if $\delta\Pi = 0$ for all $\delta\Theta$ (take $\delta\Theta_{\boxed{j}} = \begin{cases} 0 & j \neq 4 \\ 1 & j = 4 \end{cases}$) Row

we require

$$A^{(2)}_{21}\,\Theta_{\boxed{3}} + A^{(2)}_{22}\,\Theta_{\boxed{4}} + A^{(3)}_{11}\,\Theta_{\boxed{4}} + A^{(3)}_{12}\,\Theta_{\boxed{5}} = Q^{(2)}_{2} + Q^{(3)}_{1}$$

We can put this equation in a <u>system matrix</u> $[A]$ no sup. script.

$$\begin{array}{c} \\ \\ \delta\Theta_{\boxed{4}} \\ \\ \\ \end{array}
\begin{pmatrix} \ddots & & & & \\ & \ddots & & & \\ \cdots & A^{(2)}_{21} & \left(A^{(2)}_{22} + A^{(3)}_{11}\right) & A^{(3)}_{12} & \cdots \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix}
\begin{pmatrix} \Theta_{\boxed{1}} \\ \Theta_{\boxed{2}} \\ \Theta_{\boxed{3}} \\ \Theta_{\boxed{4}} \\ \Theta_{\boxed{5}} \\ \Theta_{\boxed{6}} \end{pmatrix}
=
\begin{pmatrix} \vdots \\ \vdots \\ Q^{(2)}_{2} + Q^{(3)}_{1} \\ \\ \\ \end{pmatrix}$$

$A_{\boxed{i},\boxed{j}};[A]$ stiffness matrix $\qquad \Theta_{\boxed{j}}, [\Theta] \qquad Q_{\boxed{i}}, [Q]$

The automatic procedure for forming $[A]$ is called the direct stiffness method, and denoted

$$[A] = {\sum_k}' A^k_{ij} \quad ; \quad [Q] = {\sum_k}' Q^k_i$$

as we have have seen it corresponds to summing all (2) equations corresponding to $\delta\Theta_{\boxed{i}}$ ($\rightarrow$ row $\boxed{i}$) and expressing the Θ^k_l as $\Theta_{\boxed{j}}$ (local $\rightarrow$ global). Note with this construction of $[A]$, the symmetry of the A^k_{ij} is maintained in the system matrix.

ex: take $L^k = \Delta x = \frac{1}{N-1}$ $\quad$ N : number of nodes

$\quad$ $N-1$: number of elements

so $A^k_{ij} = \frac{1}{\Delta x}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$; direct stiffness gives

$$[A] = \frac{1}{\Delta x}\begin{pmatrix} -1 & 1 & & & & \\ 1 & -1\;\;-1 & 1 & & & \\ & 1 & -1\;\;-1 & 1 & & \\ & & 1 & -1\;\;-1 & 1 & \\ & & & 1 & -1 & \\ & & & & & \ddots \end{pmatrix} = \frac{1}{\Delta x}\begin{pmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & 0 & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & 0 & & 1 & \ddots & \\ & & & & & \ddots \end{pmatrix}$$

symmetric

$$[Q] = \begin{pmatrix} \begin{bmatrix} Q_1^1 \\ Q_2^1 \end{bmatrix} + \begin{bmatrix} Q_1^2 \\ Q_2^2 \end{bmatrix} \\ \begin{bmatrix} Q_1^3 \\ Q_2^3 \end{bmatrix} + \begin{bmatrix} Q_1^4 \\ Q_2^4 \end{bmatrix} \\ \begin{bmatrix} Q_1^5 \\ \vdots \end{bmatrix} \quad \begin{bmatrix} \vdots \end{bmatrix} \end{pmatrix} = \begin{pmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_1^3 + Q_2^2 \\ Q_2^3 + Q_1^4 \\ Q_1^5 + Q_2^4 \\ \vdots \end{pmatrix}$$

6. "Systemization" of [Q]

In most of our applications we do not know $q(x)$ but rather q at the nodes; in element k we can then write

$$q^k = \sum_{j=1}^{2} q_j^k \, h_j(r) \qquad \text{(interpolate } q \text{ same as } \theta\text{)}$$

Then

$$Q_i^k = \frac{L^k}{2} \int_{-1}^{1} q \, h_i \, dr = \frac{L^k}{2} \int_{-1}^{1} h_i \sum_{j=1}^{2} q_j^k \, h_j \, dr$$

$$= B_{ij}^k \, q_j^k$$

Also denoted by M_{ij}^k

where $B_{ij}^k = \frac{L^k}{2} \int_{-1}^{1} h_i h_j \, dr$

the discrete "FEM analog" of the identity matrix.

then

$$[Q] = \sum_k{}' \, Q_i^k = \sum_k{}' \, B_{ij}^k \, q_j^k = B_{\boxed{i}\boxed{j}} \, q_{\boxed{j}}$$

q — nodal values of q in global numbering

$$[B] = B_{\boxed{i}\boxed{j}} = \sum_k{}' \, B_{ij}^k$$

for the linear basis

$$B_{ij}^k = \frac{L^k}{2}\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} = L^k \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \quad \text{(symmetric)}$$

ex: take $L^k = \Delta x = \frac{1}{N-1}$ $\quad$ N = # of nodes
$N-1$ = # of elements

$$[B] = \Delta x \begin{pmatrix} \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & \begin{bmatrix} 1/3 & 1/3 \\ 1/6 & \begin{bmatrix} 1/3 & 1/3 \\ 1/6 & \begin{bmatrix} \cdot & 1/3 \\ & \cdot \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{bmatrix} & & & \\ & 1/6 & & \\ & & 1/6 & \end{pmatrix} = \Delta x \begin{pmatrix} 1/3 & 1/6 & & & \\ 1/6 & 2/3 & 1/6 & & \\ & 1/6 & 2/3 & 1/6 & \\ & & 1/6 & \cdot & \\ & & & & \ddots \end{pmatrix}$$

(symmetric)

7. Comparison with Finite-Differences

To solve $\theta_{xx} = q \, ; \, \theta(0) = \theta(1) = 0$:

take $N-1$ equal length elements, N nodes $L^k = \Delta x = 1/_{N-1}$

Our finite element approximation is therefore (from the system matrices)

$$[A][\theta] = [B][q] \quad , \quad \text{or}$$

$$\big[[b_1]\ [b_2] \dots [b_N]\big]\begin{bmatrix} Y_1 \\ \vdots \end{bmatrix} = [f]$$

$$\sum_{i=1}^{N} Y_i [b_i] = [f]$$

$$\frac{1}{\Delta x^2}\begin{pmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & 0 & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & 0 & & 1 & -2 & 1 \\ & & & & 1 & -1 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \vdots \\ \theta_{N-1} \\ \theta_N \end{pmatrix} = \begin{pmatrix} 1/3 & 1/6 & & & & \\ 1/6 & 2/3 & 1/6 & & 0 & \\ & 1/6 & 2/3 & 1/6 & & \\ & & & \ddots & & \\ & 0 & & 1/6 & 2/3 & 1/6 \\ & & & & 1/6 & 1/3 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ \vdots \\ q_{N-1} \\ q_N \end{pmatrix}$$

where the $[\ ]$ ed system is after imposition of the Dirichlet boundary conditions by matrix condensation. (symmetric)

*** condensed or diagonal B-matrix**

$$\sum_{i=2}^{N-1} Y_i [b_i] = [f] - Y_1[b_1] - Y_N[b_N]$$

One finite element procedure consists of collapsing the

[B]-matrix onto its diagonal; $\begin{cases} B^{c}_{i,i} = \sum_{j=1}^{N} B_{i,j} \\ B_{i,j} = 0 \quad i \neq j \end{cases}$

In this particular case we recover exactly the finite-difference equations.

* <u>full B-matrix</u> (con<u>sis</u>tent approximation)

with $q = -\sin 2\pi x$ we plot $\ln \varepsilon_\infty$ vs. $\ln h$ in Fig. 3
The method is seen to be <u>second-order</u>; in this particular case the consistent approximation ≈ condensed approximation.
In general for higher-order basis and/or deformed/non-uniform meshes the consistent approximation will give better accuracy.

N.B. the finite element scheme does <u>not</u> give the <u>compact scheme</u>

8. Natural / Internal Boundaries

<u>* natural cond's</u>

$\Pi : \int_0^1 \left\{ -\frac{1}{2}\left(\frac{d\theta}{dx}\right)^2 - q\,\theta \right\} dx$; $\delta\Pi = 0$ with <u>no restriction</u> on $\delta\theta(1)$ (but $\delta\theta(0) = 0$) gives:

$\frac{d^2\theta}{dx^2} = q \qquad \theta(0) = 0 \qquad \frac{d\theta}{dx}(1) = 0$

natural <u>boundary</u> condition

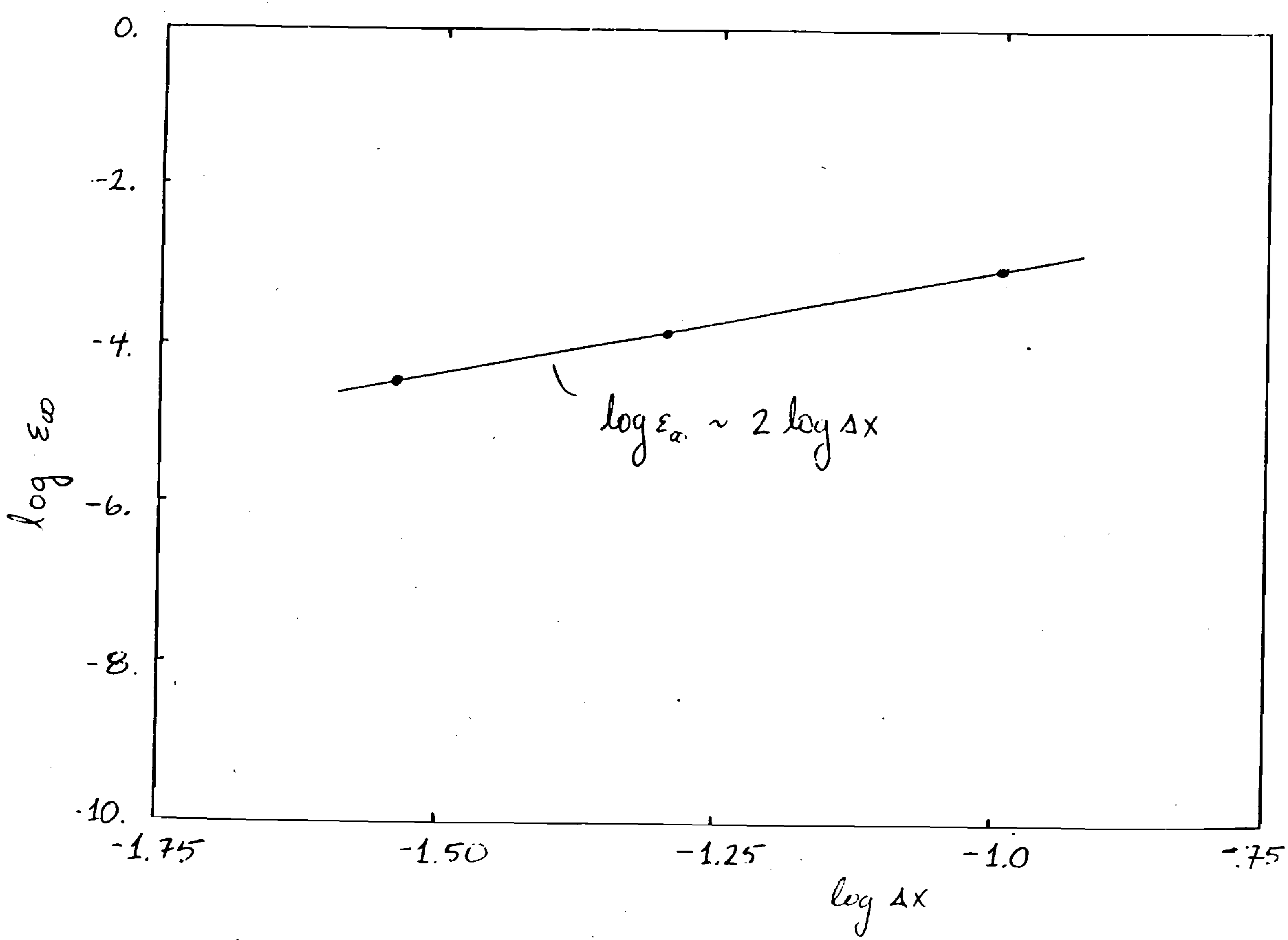


FIG 3 LINEAR FINITE-ELEMENT SCHEME

$$u_{xx} = \sin(2\pi x)$$

$$u(0) = u(1) = 0$$

Likewise, in the finite element procedure, if we "do nothing" at $x=1$, we get the equations

$$\frac{1}{\Delta x}\begin{pmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -1 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \vdots \\ \theta_{N-1} \\ \theta_N \end{pmatrix} = \Delta x \begin{pmatrix} 1/3 & 1/6 & & & & \\ 1/6 & 2/3 & 1/6 & & & \\ & 1/6 & 2/3 & 1/6 & & \\ & & & \ddots & & \\ & & & 1/6 & 2/3 & 1/6 \\ & & & & 1/6 & 1/3 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_{N-1} \\ q_N \end{pmatrix}$$

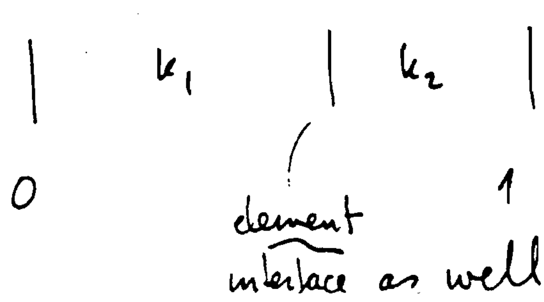
the last equation reads $\dfrac{\theta_{N-1}-\theta_N}{\Delta x} = O(\Delta x)$

$$\Rightarrow \quad \frac{d\theta}{dx}(x=1) = \text{"}0\text{"} + O(\Delta x)$$

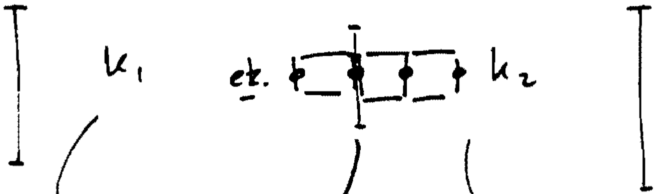
in other words the equations "naturally" impose a first-order zero-derivative boundary condition. {The solution in the interior "typically" remains 2nd-order}. In HW1 asked to impose non-ze[ro] gradient condition; modify q.

* <u>internal boundaries, differing conductivity</u>

if we have a domain



(14)



$$k_1 \frac{d^2\theta_1}{dx^2} = q_1 \qquad\qquad k_2 \frac{d^2\theta}{dx^2} = q_2$$

$$\theta_1 = \theta_2 \qquad \text{temperature}$$

$$k_1 \frac{d\theta_1}{dx} = k_2 \frac{d\theta_2}{dx} \qquad \text{flux}$$

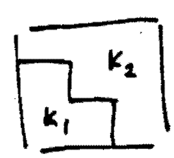
The formal direct stiffness procedure will automatically give continuity of (θ by construction, and) flux (natural boundary condition); in finite difference procedure, flux continuity equations must be explicitly supplied.

8. Advantages of Finite-Elements

* for ∇^2, symmetric, ± definite for general boundary conditions

* general geometry : • local co-ordinate system / direct stiffness
  • integral method : boundary conditions "easy"

* non-homogeneities easily handled :



* systematically extends to higher order