LECTURE 2: INTRO TO MATHEMATICAL PROGRAMMING

Today: Introducing some lingo that we'll use throughout the course

1. Terminology

Example: Suppose you're the CEO of Peyamazon, a very successful company that only sells 5 kinds of goods:

- Apples (A)
- Books (B)
- Coconuts (C)
- Donuts (D)
- Electronics (E)

Decision Variables: The variables whose values are under our control and influence the performance of our system. Here they are just A, B, C, D, E

Suppose A costs \$2, B costs \$3, C costs \$5, D costs \$2 and E costs \$7, but the catch is that instead of selling E, you're giving them away.

Objective Function: The function you would like to maximize or minimize. Here we want to find the values of A, B, C, D, E to maximize total profit, which is

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$$2A + 3B + 5C + 2D - 7E$$

Constraints: Restrictions on the values of decision variables. In most situations, only certain values of the decision variables are possible.

Of course, in this scenario, all the variables are non-negative. But suppose for example that Peyamazon:

- Can only get most 500 donuts
- Can only get at most 1000 pieces of produce (Apples and Coconuts)
- Must order exactly 300 pieces of Books and Electronics (from a parent manufacturing company)
- Must order at least 100 books

Those constraints can be represented mathematically as

$$\begin{aligned} A, B, C, D, E \geq& 0 \\ D \leq& 500 \\ A + C \leq& 1000 \\ B + E =& 300 \\ B \geq& \ 100 \end{aligned}$$

The Complete Optimization Problem Our goal, which is to

$$\text{Maximize } z = 2A + 3B + 5C + 2D - 7E$$

Subject to (s.t.) the constraints below.

Domain/Feasible Region: The domain where our variables lie in, where the constraint is satisfied.

For example, the following is in the feasible region

$$A = 200, B = 200, C = 500, D = 300, E = 100$$

But the following is not

$$A = 200, B = 500, C = 1000, D = 100, E = 100$$

Here the constrains $A + C \leq 1000$ and $B + E = 300$ are violated.

Note: In general, the feasible region is a complicated object (think a 3D polygon), but its geometry will play an important role in the course.

Optimal Solution: Any point(s) in the domain that optimizes (here: maximizes) the objective function. May or may not be unique. For example, the optimal solution in this model is

$$A = 0, B = 300, C = 1000, D = 500, E = 0$$

$$z = 2(0) + 3(300) + 5(1000) + 2(500) - 7(0) = \$6900$$

(This is not too difficult to obtain, since E counts negatively towards our profit, and also C is more valuable than A)

In summary: Operations Research means representing a real-world system by a mathematical model, and then solving such a model using algorithms. The mathematical model usually consists of

- Decision Variables
- Constraints (to represent the physical limitations of the system)

- An objective function
- An optimal solution

2. Interlude: Societal Impact

The tools we learn in this course are very powerful, but also very dangerous, and they come at a price.

Notice for instance in the above example, that to maximize profits, we had to set $E = 0$, meaning giving away no electronics at all, which was literally the only perk a customer would get!

Societal Costs: And in fact the profits that companies make usually come with a societal cost, affecting employees and customers as well. If you're interested more about those adverse effects, I highly recommend you read Weapons of Math Destruction by Cathy O'Neil, which talks more about the societal impacts of algorithms.

Taco Bell: Last time we talked about taco bell being able to increase its profits by "optimizing worker schedules," but there is no mention of how it affected the employees. Did they do this by reducing staffing or the quality of life for staff in general?

Police use of data analytics: This example is mentioned in the book above, but police departments used algorithms to localize high areas of crime. On first glance, this seems a good strategy to optimize government resources and police funding. And the results support this: more arrests are made with fewer staff on payroll.

But what we don't see here is that the use of data here is racially biased, in particular towards black men. As a result, more black men are arrested, crime statistics in black neighborhoods shoot up, more police officers are sent to those neighborhoods, and the pattern escalates, eventually contributing to racial disparity in incarceration rates, etc.

The moral of the story is: Please use those models with caution, thinking about the impact on society as a whole.

Possible Solutions: That said, the math is not all evil, you *can* use it for good. For example, you can modify the objective function so as to reward good acts. For example, just change the $-7E$ in the example above to $+7E$. It won't correspond to your actual profit, but you'll incentivize making the customer happy by giving away electronics. Or instead of maximizing profits, try to maximize "profit + employee satisfaction." This is sometimes called a discounting factor.

3. Mathematical Programming

In general, our problem takes the following form:

Decision Variables: $x_1, x_2, \cdots x_n$ (before we used A, B, C, D, E)

Objective Function:

$$z = f(x_1, \cdots, x_n)$$

In our previous example, we had

$$z = 2x_1 + 3x_2 + 5x_3 + 2x_4 - 7x_5$$

f could be nonlinear, so in theory could have a crazy function like

$$f(x_1, x_2, x_3) = (x_1)^2 - 2x_2 + (x_1)(x_3)$$

Constraints: Usually $x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0$ and

$$\begin{aligned} g_1(x_1, \cdots, x_n) &\leq b_1 \\ g_2(x_1, \cdots, x_n) &\leq b_2 \\ &\vdots \\ g_k(x_1, \cdots, x_n) &\leq b_k \end{aligned}$$

In our previous example, the constraints were $x_i \geq 0$ and

$$\begin{aligned} x_4 &\leq 500 \\ x_1 + x_3 &\leq 1000 \\ x_2 + x_5 &\leq 300 \\ -(x_2 + x_5) &\leq -300 \\ -x_2 &\leq -100 \end{aligned}$$

(The third and fourth line give you $x_2 + x_5 = 300$ and the last line gives you $x_2 \geq 100$)

Optimization Problem

$$\max z = f(x_1, \cdots, x_n) \text{ subject to constraints above}$$

Usually f will be a "nice" function like convex or linear, and usually the variables will have a certain form, like $x_i \in \mathbb{R}$ or $x_i \in \mathbb{Z}$

4. Linear Programming

If f and the constraints are linear, then we call this linear programming.

Linear just means of the form $c_1 x_1 + \cdots c_n x_n$

Example:

$$\max z = 2x_1 - x_2 + x_3 - 4x_4$$

Subject to $x_i \geq 0$ and

$$\begin{aligned} 2x_1 - 4x_2 + 3x_3 - x_4 &\leq 5 \\ x_2 + x_4 &\leq -3 \\ x_1 + x_3 + 2x_4 &\leq 8 \end{aligned}$$

Matrix Form: The objective function can be written as a dot product:

$$2x_1 - x_2 + x_3 - 4x_4 = \begin{bmatrix} 2 & -1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -4 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c^T \mathbf{x}$$

Where $c = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -4 \end{bmatrix}$ (the coefficients) and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ (the variables)

Moreover, the constraint can be rewritten as

$$\begin{bmatrix} 2 & -4 & 3 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} 5 \\ -3 \\ 8 \end{bmatrix}$$

(Here we mean $\leq$ component-wise, that is means each component on the left is $\leq$ to the component on the right)

So if we let A to be the matrix of coefficients and b be the numbers on the right hand side then the constraint becomes $A\mathbf{x} \leq b$

Summary: Any linear programming problem can be written as

$$\begin{gathered}\max c^T \mathbf{x}\\ \text{Subject to } A\mathbf{x} \leq b\\ \text{and } \mathbf{x} \geq 0\end{gathered}$$

c is sometimes called the cost vector and b the constraint vector

5. Integer Linear Programming

If the x_i are integers, then this is called integer linear programming. Here's a neat example:

Example: Class Assignments

Suppose you're the provost of Brown and you're trying to assign students to classes.

You are given a matrix P of preferences, where

$$p_{ij} = \begin{cases} 1 & \text{if student } i \text{ wants to take class } j \\ 0 & \text{otherwise} \end{cases}$$

For example, if

$$P = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This means the first student wants to take classes 1, 3 and 4, the second student wants to take 1, 2 and 4, and the third student only class 3.

Your goal: Define the decision variables, constraints, and objective function

Decision Variables: $x_{ij} \in \{0, 1\}$ (integer programming)

Where 1 means student i is placed in class j

And 0 means student i is not placed in class j

A (possibly non-optimal) example to keep in mind is

$$\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Objective Function: You would like to maximize the total number of assignments, that is

$$\max z = \sum_{i,j} x_{ij}$$

In the example above, we have $z = 5$

(This makes sense if you think of each student paying \$1000 for each class; notice again the societal impact)

Constraint: First of all, we would like to satisfy the preferences of each student, so $\mathbf{x} \leq \mathbf{p}$, that is $x_{ij} \leq p_{ij}$ for each i and j. In fact, in the example above, notice how $\mathbf{x}$ is kind of a subset of $\mathbf{p}$

Second, each student needs to be enrolled in at least one class, so

$$\sum_{j} x_{ij} \geq 1$$

(For each row, you sum over the columns, since for each student you sum over the classes). Here this just means that there are no 0 rows.

Third, each student can only take 4 classes at most, so actually we also have

$$\sum_j x_{ij} \leq 4$$

Finally, each class has a cap, so the number of students in class j must be smaller than the cap b_j that is

$$\sum_i x_{ij} \leq b_j$$

(For each column, you sum over all the rows, since for each class you sum over the students)

And so our problem becomes

$$\max z = \sum_{i,j} x_{ij}$$

Subject to

$$\begin{aligned} \mathbf{x} &\leq \mathbf{p} \\ -\sum_j x_{ij} &\leq -1 \\ \sum_j x_{ij} &\leq 4 \\ \sum_i x_{ij} &\leq b_j \end{aligned}$$

Which is relatively easy to solve if you only have 3 students and 4 classes, but much harder for bigger classes.