LECTURE 5: EXTREME POINTS OF FEASIBLE REGIONS + SIMPLEX ALGORITHM (I)

1. Vertices

We are now ready to give the algebraic definition of a vertex, one that is programmable with a computer.

Example 1:

$$\begin{aligned} \max z &= \text{Blah} \\ \text{subject to } x_1 + 2x_2 &\leq 5 \\ x_1 - 3x_2 &\leq 7 \\ x_2 + x_3 &= 4 \\ x_1 + 3x_3 &= 1 \end{aligned}$$

The constraints with " $=$ " (here 3 and 4) are called **equality constraints**, the others have " $\leq$ " It *could* happen that *at a particular point*, the " $\leq$ " becomes " $=$," we call that **active/binding**

Definition/Example: If for some point $y = (y_1, y_2, y_3)$, we happen to have $y_1 - 3y_2 = 7$, then the second constraint is **active/binding** at y

Linear Independence: Some constraints can be redundant. For example, if we consider

$$\begin{cases} x_1 + 3x_2 + 4x_3 \leq 6 \\ 2x_1 + 6x_2 + 8x_3 \leq 12 \end{cases}$$

Date: Tuesday, September 20, 2022.

Then we can safely ignore the second constraint. This is an example of a **linearly dependent** constraint. This just means that $\begin{bmatrix}1\\3\\4\end{bmatrix}$ and $\begin{bmatrix}2\\6\\8\end{bmatrix}$ are linearly dependent (in the linear algebra sense).

But a constraint like

$$\begin{cases} x_1 + 3x_2 + x_3 \leq 6 \\ 2x_1 + 5x_2 + 2x_3 \leq 7 \end{cases}$$

is linearly independent; each constraint gives us nontrivial information.

Definition: We say y is a **basic solution** if

(1) All the " $=$ " constraints are satisfied

(2) There are a total of n (linearly ind.) active constraints at y

With regards to the example above, $y = (y_1, y_2, y_3)$ would be a basic solution if we had

$$y_2 + y_3 = 4$$
$$y_1 + 3y_3 = 1$$

And <u>either</u> $y_1 + 2y_2 = 5$ <u>or</u> $y_1 - 3y_2 = 7$ to get a total of 3 constraints with "="

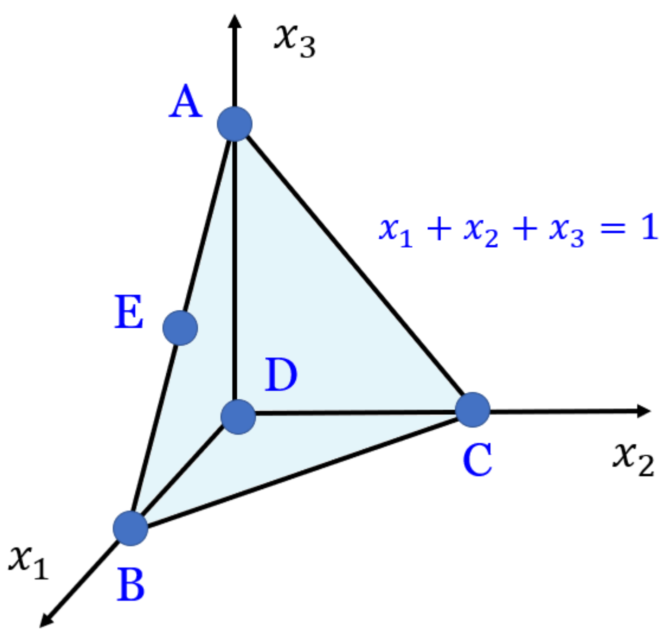
Definition 3: [Vertex/Basic Feasible Solution]

A **vertex/basic feasible solution** is just a basic solution that is in the feasible region.

Example: Consider the following constraint

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \end{cases}$$

Then in the picture below, A, B, and C are basic solutions.



For example, at A, we have

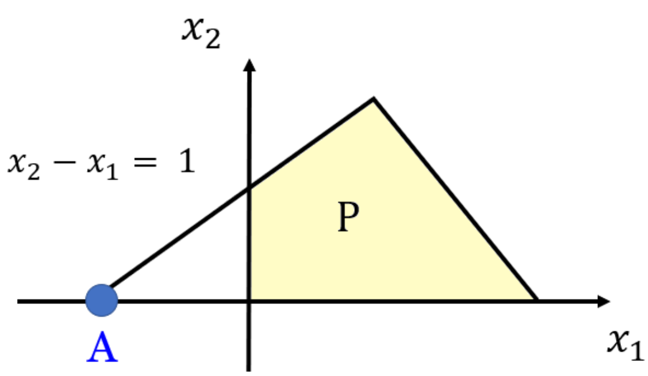
$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 = 0 \\ x_2 = 0 \end{cases}$$

So the equality constraint is satisfied, and there are 3 constraints with equality in total.

D is not basic because the equality constraint $x_1 + x_2 + x_3 = 1$ is not satisfied.

E is not basic because there are just two constraints with equality, namely $x_1 + x_2 + x_3 = 1$ and $x_2 = 0$

Example: Here is an example of a basic solution that is not feasible:

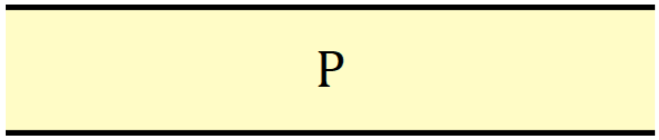


The point A is not in the feasible region P, but it is basic because two constraints are satisfied with equality, $x_2 = 0$ and $x_2 - x_1 = 1$

2. Existence of Vertices

Question: When does a polyhedron have at least one vertex?

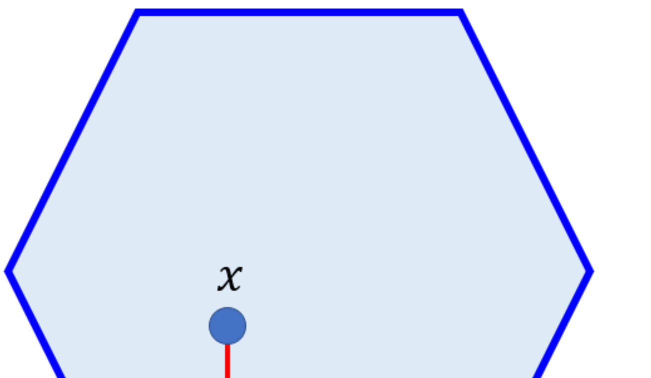
Non-Example: The following region between two parallel lines has no vertex



It turns out that this is the only bad scenario that can happen. In fact:

Fact: If P is a polyhedron that doesn't contain an infinite line, then P has at least one vertex

Idea of Proof: Start with a point $\mathbf{x}$, then move along a direction along which all active constraints remain active, until you hit a new constraint.



In that case, the total number of = constraints increased by at least 1. Then repeat this (move along a new direction to increase the constraints), until you reach n (linearly independent) active constraints, at which point we reached a vertex.

In particular, if P is bounded, it doesn't contain a line. And also the positive orthant $\{\mathbf{x} \geq \mathbf{0}\}$ (the one we usually deal with) also doesn't contain a line. Therefore:

Fact: Every nonempty bounded polyhedron or every nonempty polyhedron in standard form has at least one vertex (= basic feasible solution)

3. Optimality and Extreme Points

We are finally ready to prove that the max/min has to be attained at a vertex, a fact that we observed a couple of lectures ago.

LP Problem:

$$\begin{aligned} \max z = & c^T x \\ \text{Suject to } & Ax \leq b \end{aligned}$$

Theorem: If a LP problem has a finite optimal solution, then that solution must be attained at a vertex.

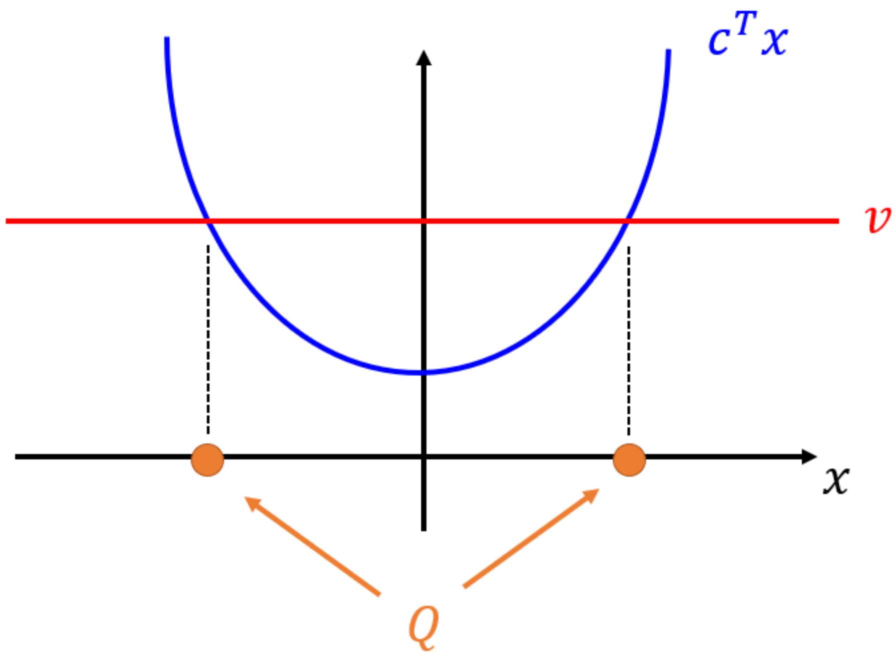
Remarks: That solution *could* be attained inside the polyhedron as well. (Think for example when z is always 1; in that case every point is a solution).

We're not saying *yet* that the solution exists, but more on that later.

Proof: Let P be the polyhedron $Ax \leq b$.

STEP 1: Let v be that optimal value, and consider the set of points where that value is attained. More precisely, let

$$Q =: \left\{ x \in \mathbb{R}^n \mid Ax \leq b \text{ and } c^T x = v \right\}$$



Then Q is also a polyhedron, since you're just adding an extra constraint $c^T x = v$ to P

Moreover $Q \subseteq P$. Hence, since P doesn't contain a line, Q doesn't contain a line.

Therefore, by the fact above, Q has at least one vertex, say $w \in Q$.

STEP 2:

Claim: w is a vertex of P

Then we would be done because max $c^T x$ would be attained at a vertex, namely w

Proof of Claim:

Recall: x is an **extreme point** if $x = \lambda y + (1-\lambda)z \Rightarrow x = y$ or $x = z$

Suppose $w = \lambda y + (1-\lambda)z$ with $y, z \in P$ and $0 \leq \lambda \leq 1$, want to show $w = y$ or $w = z$

Since $v = \max c^T x$, we have $c^T y \leq v$ and $c^T z \leq v$.

If either $c^T y < v$ or $c^T z < v$ we get a contradiction because then

$$v = c^T w = c^T (\lambda y + (1-\lambda)z) = \lambda \left(c^T y\right) + (1-\lambda)\left(c^T z\right) < \lambda v + (1-\lambda)v = v$$

Hence $c^T y = v$ and $c^T z = v$ which implies $y, z \in Q$

But then since w is an extreme point for Q and $w = \lambda y + (1-\lambda)z$ for $y, z \in Q$ we get $w = y$ or $w = z$, which is what we wanted to show. $\square$

A priori, it the optimal solution might not exist, but good news is that in fact it does:

Fact: If P has at least one vertex, then either the optimal cost is $z = \infty$ or there is an optimal vertex.

(The proof is similar to the one about existence of vertices)

And combined with the fact that polyhedra have vertices, we finally get our glorious result:

Glorious Result: If P is a nonempty polyhedron, then either $z = \infty$ or there is an optimal solution, which is attained at a vertex.

In particular, if P is bounded, then z cannot be ∞, and there must be an optimal solution at a vertex.

4. The Simplex Algorithm

In theory the LP problem is super easy now: Just check all the vertices and then you're done.

In practice it's more complicated: There might be lots of vertices or you could have a weird $27-$dimensional polygon!

What we'll explore now is the **simplex method**, which is a faster way of finding an optimal vertex, without even having to look at all of them.

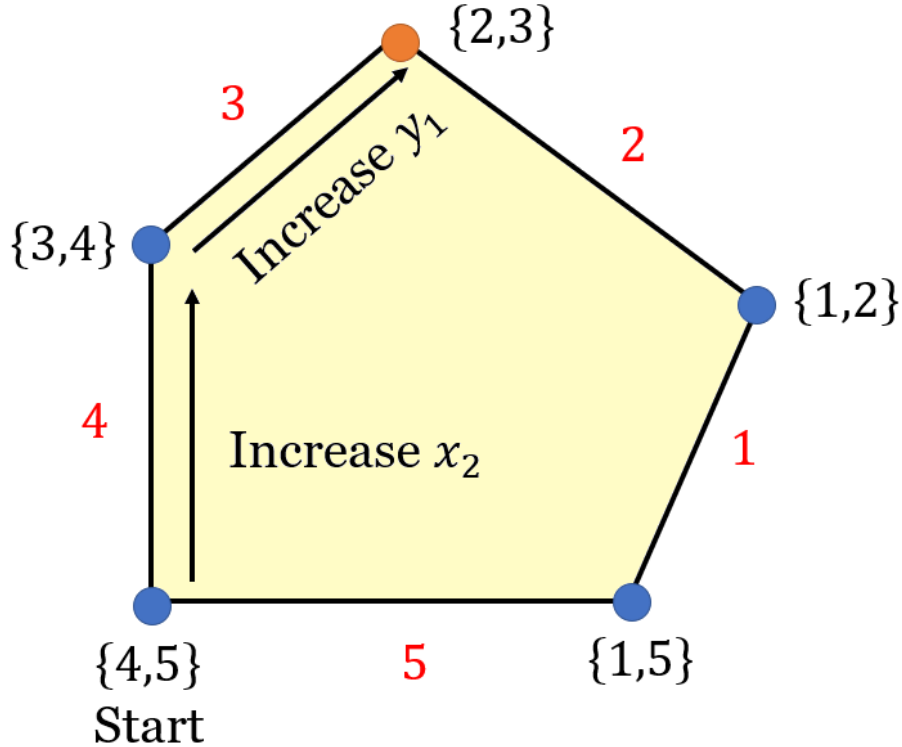
Main Idea: Start at a vertex, and then move to a neighboring vertex that gives you a bigger value. Stop if all your neighbors give you smaller values.

Let's go over the details with a specific example:

Example 2:

$$
\begin{aligned}
\max z = & 2x_1 + 5x_2 \\
\text{subject to } & 2x_1 - x_2 \leq 4 & (1) \\
& x_1 + 2x_2 \leq 9 & (2) \\
& -x_1 + x_2 \leq 3 & (3) \\
& x_1 \geq 0 & (4) \\
& x_2 \geq 0 & (5)
\end{aligned}
$$

Optional: Here is a picture of the feasible region:



STEP 1: Start at a vertex, say $(0, 0)$

Notice ④ and ⑤ are satisfied with equality, we call this **tight** and the $z-$value is 0.

Current Vertex: {④, ⑤}

Objective Value: 0

$z = 0$ is not optimal: Since $z = 2x_1 + 5x_2$, and $2 > 0$ and $5 > 0$, we can increase x_1 or x_2 to increase the value of z. We choose x_2 since 5 is the biggest number (= fastest increase)

Move: Increase x_2 until you hit a new constraint.

Then ⑤ is **released**, meaning it's not tight any more and ③ becomes tight. In that case stop at $x_2 = 3$.

Note: ③ is hit first, since ① is decreasing with x_2, and ② stops at $x_2 = \frac{9}{2} = 4.5$, which is after $x_2 = 3$.

Then you hit the new vertex {④, ③}, which is $(0, 3)$

(Continued in the next lecture)