LECTURE 6: SIMPLEX ALGORITHM (II)

Today: Examples of the Simplex Algorithm, which is an efficient way of finding the optimal vertex.

1. Simplex Algorithm Example

Example 1:

$$\begin{aligned} \max z =& 2x_1 + 5x_2 \\ \text{subject to } & 2x_1 - x_2 \leq 4 && \text{①} \\ & x_1 + 2x_2 \leq 9 && \text{②} \\ & -x_1 + x_2 \leq 3 && \text{③} \\ & x_1 \geq 0 && \text{④} \\ & x_2 \geq 0 && \text{⑤} \end{aligned}$$

Optional Picture: See next page

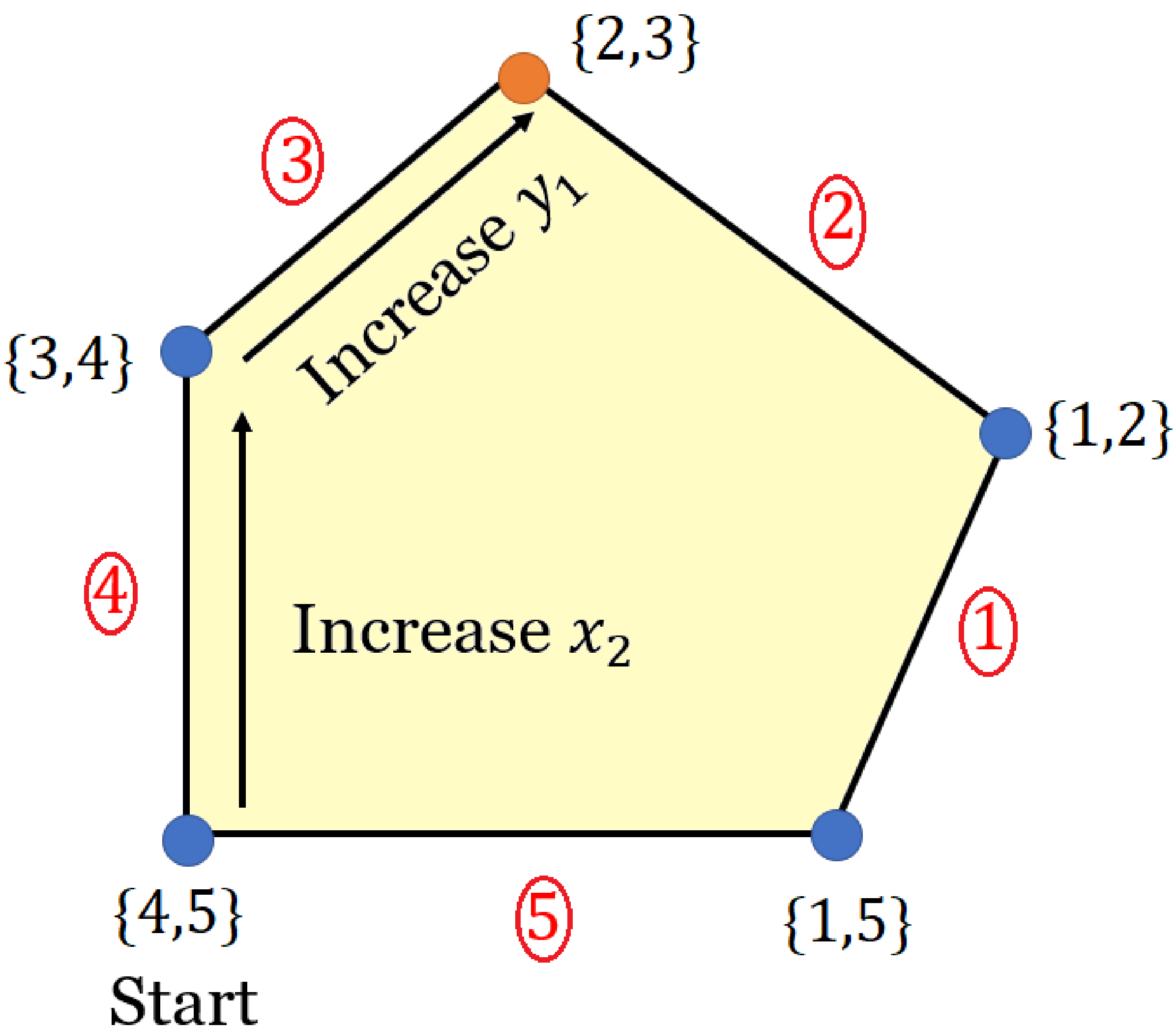
STEP 1: Start at $(0, 0)$

Current Vertex: {④, ⑤} (the constraints satisfied with equality)

Objective Value: 0

Not optimal! we can increase x_1 or x_2 to increase the value of z. We choose x_2 since 5 is the biggest number (= fastest increase)

Date: Tuesday, September 27, 2022.



Move: Increase x_2 until you hit a constraint

Question: Which constraint ①, ②, ③ is hit first?

We are moving along the line $(0, x_2)$, so let's set $x_1 = 0$ in each constraint and see which x_2 value is obtained:

$$
\begin{aligned}
&① \qquad 2(0) - x_2 = 4 \Rightarrow x_2 = -4 \ \times \\
&② \qquad 0 + 2x_2 = 9 \Rightarrow x_2 = \frac{9}{2} = 4.5 \\
&③ \qquad -0 + x_2 = 3 \Rightarrow x_2 = 3
\end{aligned}
$$

③ is hit first since it gives the smallest hitting time $x_2 = 3$

So you hit the new vertex $\{④, ③\} = (0, 3)$

Note: What helped us in this problem is that we started at the origin and our constraints were $x_1 \geq 0$ and $x_2 \geq 0$.

Idea: Introduce new coordinates (y_1, y_2) so that $(0, 3)$ becomes the new origin and the constraints ④ and ③ become $y_1 \geq 0$ and $y_2 \geq 0$

Fact:

If the constraint is $a^T x \leq b$, then $y = b - a^T x$

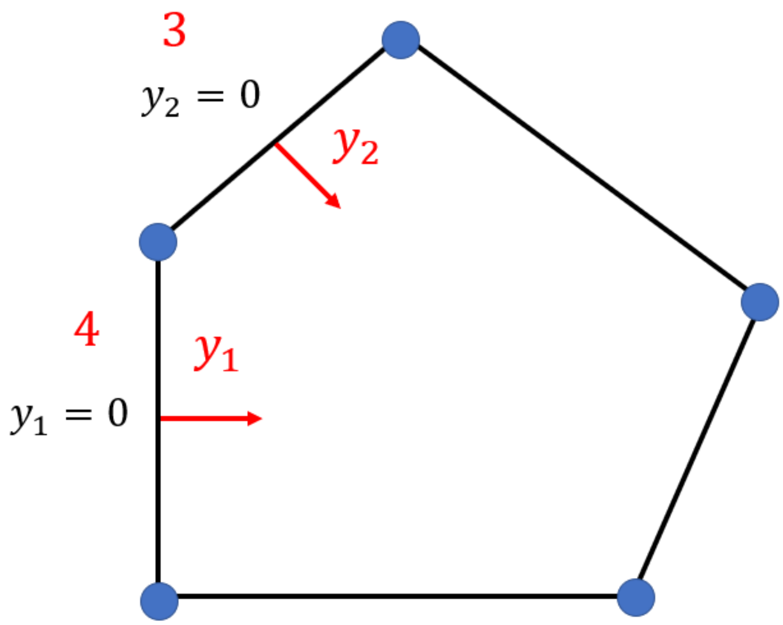
(Think $b - a^T x \geq 0$, making the variables positive, like slack variables)

$$
\begin{aligned}
&④ \qquad x_1 \geq 0 \Rightarrow y_1 = x_1 \\
&③ \qquad -x_1 + x_2 \leq 3 \Rightarrow 3 + x_1 - x_2 \geq 0 \Rightarrow y_2 = 3 + x_1 - x_2
\end{aligned}
$$

Coordinates:

$$
\begin{cases}
y_1 = x_1 \\
y_2 = 3 + x_1 - x_2
\end{cases}
$$

Geometric Interpretation: Here y_1 and y_2 are the directions perpendicular to the constraints ④ and ③, as in the following picture:



Rewrite LP in terms of y_1 and y_2:

$$\begin{cases} x_1 = y_1 \\ y_2 = 3 + x_1 - x_2 \Rightarrow x_2 = 3 + x_1 - y_2 = 3 + y_1 - y_2 \end{cases}$$

$$z = 2x_1 + 5x_2 = 2y_1 + 5(3 + y_1 - y_2) = 15 + 7y_1 - 5y_2$$

And similarly, you rewrite the constraints, and so the LP becomes

$$\max z = 15 + 7y_1 - 5y_2$$

$$\begin{aligned} \text{Subject to } & y_1 + y_2 \leq 7 && (1) \\ & 3y_1 - 2y_2 \leq 3 && (2) \\ & y_2 \geq 0 && (3) \\ & y_1 \geq 0 && (4) \\ & -y_1 + y_2 \leq 3 && (5) \end{aligned}$$

STEP 2: Repeat, but for the new vertex

Current Vertex: $\{④, ③\}$

Objective Value: $z = 15$

Not optimal, because of the 7 in $7y_1$

Move: Increase y_1 (so ④ is released)

If $y_2 = 0$ then

$$\begin{array}{ll} ① & y_1 + 0 = 7 \Rightarrow y_1 = 7 \\ ② & 3y_1 - 0 = 3 \Rightarrow y_1 = 1 \\ ⑤ & -y_1 + 0 = 3 \Rightarrow y_1 = -3 \times \end{array}$$

So we hit the constraint ② first

Coordinates: Vertex $\{②, ③\} = (1, 0)$ has coordinates (z_1, z_2), where:

$$\begin{cases} z_1 = 3 - (3y_1 - 2y_2) = 3 - 3y_1 + 2y_2 \\ z_2 = y_2 \end{cases}$$

Rewrite LP:

$y_2 = z_2$

$z_1 = 3 - 3y_1 + 2y_2 \Rightarrow 3y_1 = 3 - z_1 + 2y_2 = 3 - z_1 + 2z_2 \Rightarrow y_1 = 1 - \frac{1}{3}z_1 + \frac{2}{3}z_2$

$$\begin{cases} y_1 = 1 - \frac{1}{3}z_1 + \frac{2}{3}z_2 \\ y_2 = z_2 \end{cases}$$

Then in terms of our new variables, the LP problem becomes

$$\max z = 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$$

$$\text{Subject to } -\frac{1}{3}z_1 + \frac{5}{3}z_2 \le 6 \qquad \textcircled{1}$$

$$z_1 \ge 0 \qquad \textcircled{2}$$

$$z_2 \ge 0 \qquad \textcircled{3}$$

$$\frac{1}{3}z_1 - \frac{2}{3}z_2 \le 1 \qquad \textcircled{4}$$

$$\frac{1}{3}z_1 + \frac{1}{3}z_2 \le 4 \qquad \textcircled{5}$$

Note: Strictly speaking, it's not necessary here to rewrite the constraints here in terms of z_1 and z_2 because we'll see below that $z = 22$ is optimal.

STEP 3:

Current Vertex: $\{\textcircled{2}, \textcircled{3}\}$

Objective Value: 22

This *is* optimal because all the coefficients of z_1 and z_2 are negative!

That is, no matter which direction you move in, you will only *decrease* the value of z (remember $z_1 \ge 0$ and $z_2 \ge 0$)

And so the maximal value is 22

Optimal Vertex: The optimal vertex is $(0, 0)$ in $z-$coordinates

Now use our formulas for y_1, y_2 and x_1, x_2 to write it back in $x-$coordinates

$$\begin{cases} z_1 = 0 \\ z_2 = 0 \end{cases}$$

$$\begin{cases} y_1 = 1 - \frac{1}{3}z_1 + \frac{2}{3}z_2 = 1 - 0 + 0 = 1 \\ y_2 = z_2 = 0 \end{cases}$$

$$\begin{cases} x_1 = y_1 = 1 \\ x_2 = 3 + y_1 - y_2 = 3 + 1 - 0 = 4 \end{cases}$$

So our optimal vertex is $(x_1, x_2) = (1, 4)$ and our optimal value is $z = 22$.

2. Higher Dimensional Example

This method is so powerful that we can apply this to 3+ variables

Example 2:

$$\begin{aligned} \max z = & 3x_1 + 4x_2 + 4x_5 \\ \text{subject to } & x_1 + x_3 \leq 4 \qquad \text{①} \\ & 2x_2 + x_4 \leq 12 \qquad \text{②} \\ & 3x_1 + 2x_2 + x_5 \leq 18 \qquad \text{③} \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \qquad \text{④} - \text{⑧} \end{aligned}$$

(Will only do **STEP 1**, just to show you how it works)

STEP 1: Start at $(0, 0, 0, 0, 0)$

Current Vertex: $\{④, ⑤, ⑥, ⑦, ⑧\}$

Objective Value: $z = 0$

Not optimal, because of the 3, 4, and 4 coefficients

We can increase either x_2 or x_5, let's do it for x_2

Move: Increase x_2 (so ⑤ is released)

If x_1, x_3, x_4, and x_5 are 0 then

$$
\begin{array}{ll}
② & 2x_2 + 0 = 12 \Rightarrow x_2 = 6 \\
③ & 3(0) + 2x_2 + 0 = 18 \Rightarrow x_2 = 9
\end{array}
$$

So we hit the constraint ② first

Coordinates: Vertex $\{②, ④, ⑥, ⑦, ⑧\}$ has coordinates $(y_1, y_2, y_3, y_4, y_5)$

$$
\begin{cases}
y_1 = x_1 \\
y_2 = 12 - 2x_2 - x_4 \\
y_3 = x_3 \\
y_4 = x_4 \\
y_5 = x_5
\end{cases}
$$

Rewrite LP:

$$
\begin{cases}
x_1 = y_1 \\
2x_2 = 12 - y_2 - x_4 \Rightarrow x_2 = 6 - \frac{1}{2}y_2 - \frac{1}{2}y_4 \\
x_3 = y_3 \\
x_4 = y_4 \\
x_5 = y_5
\end{cases}
$$

$$z = 3x_1 + 4x_2 + 5x_5 = 3y_1 + 4\left(6 - \frac{1}{2}y_2 - \frac{1}{2}y_4\right) + 5y_5 = 24 + 3y_1 - 2y_2 - 2y_4 + 4y_5$$

And similarly for the constraints

Then in terms of our new variables, the LP problem becomes

$$\begin{aligned} \max\, & z = 24 + 3y_1 - 2y_2 - 2y_4 + 4y_5 & \\ \text{subject to}\, & y_1 + y_3 \leq 4 & \textcircled{1} \\ & y_2 \geq 0 & \textcircled{2} \\ & 3y_1 - y_2 - y_4 + y_5 \leq 6 & \textcircled{3} \\ & y_1 \geq 0 & \textcircled{4} \\ & 6 - \frac{1}{2}y_2 - \frac{1}{2}y_4 \geq 0 & \textcircled{5} \\ & y_3, y_4, y_5 \geq 0 & \textcircled{6} - \textcircled{8} \end{aligned}$$

STEP 2: Then you would continue, increase y_1 because this gives you a positive coefficient, then release ③ until you hit constraint ③ with $y_1 = 2$, change coordinates, check if this is optimal, etc.

Note: Had you started by increasing x_5, you would have immediately obtained the optimal vertex $(0, 0, 0, 0, 18)$ with optimal value $z = 72$. You can actually guess x_5 beforehand because it would give you a bigger hitting time $x_5 = 18$ compared to $x_2 = 6$.

3. MATLAB implementation

The cool thing is that you won't ever have to perform the simplex algorithm by hand (except for the homework and the exam) because there is a package on MATLAB that does it for you!

Example 3:

$$\begin{aligned}\max z =& 2x_1 + 5x_2 \\ \text{subject to } & 2x_1 - x_2 \leq 4 \\ & x_1 + 2x_2 \leq 9 \\ & -x_1 + x_2 \leq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0\end{aligned}$$

In order to input this into MATLAB, we need to write this in the form $z = 0$ and $Ax \leq b$:

$$\begin{aligned}\max\ & z - 2x_1 - 5x_2 = 0 \\ \text{subject to } & 2x_1 - x_2 \leq 4 \\ & x_1 + 2x_2 \leq 9 \\ & -x_1 + x_2 \leq 3 \\ & -x_1 \leq 0 \\ & -x_2 \leq 0\end{aligned}$$

$$f = \begin{bmatrix} -2 & -5 \end{bmatrix} \text{ Coefficients of } z\text{, row vector}$$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ -1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = \begin{bmatrix} 4 \\ 9 \\ 3 \\ 0 \\ 0 \end{bmatrix} \text{ Column vector}$$

```
f = [-2,-5]
A = [2 -1;1 2; -1 1; -1 0; 0 -1]
b = [4 9 3 0 0]'
x = linprog(f,A,b)
```

Then gives you $x = (1, 4)$, as we found above. If you want the optimal value, you just type

```
[x,z] = linprog(f,A,b)
```

This gives you $x = (1, 4)$ and $z = -22$, and then the optimal value is $z = -(-22) = 22$

Note: We have to put that minus-sign because MATLAB solves $\min z$ instead of $\max z$. The two are related via the equation

$$\max z = -\min(-z)$$

This is also why we have to input -2 and -5 in f instead of 2 and 5

Note: Similarly, if you apply this to the previous example, you would get $x = (0, 0, 0, 0, 18)$ and $z = -72$, which means the max is 72.