1. 20. From past data, the production manager of a factory knows that, by varying his production rate, he incurs additional costs. He estimates that his cost per unit increases by \$0.50 when production is increased from one month to the next. Similarly, reducing production increases costs by \$0.25 per unit. A smooth production rate is obviously desirable.

Sales forecasts for the next twelve months are (in thousands):

July	4	October	12	January	20	April	6
August	6	November	16	February	12	May	4
September	8	December	20	March	8	June	4

June's production schedule already has been set at 4000 units, and the July 1 inventory level is projected to be 2000 units. Storage is available for only 10,000 units at any one time. Ignoring inventory costs, formulate a production schedule for the coming year that will minimize the cost of changing production rates while meeting all sales demands. (*Hint*: Express the change in production from month t to month $t+1$ in terms of nonnegative variables x_t^+ and x_t^- as $x_t^+ - x_t^-$. Variable x_t^+ is the increase in production and x_t^- the decrease. It is possible for both x_t^+ and x_t^- to be positive in the optimal solution?)

Let x_t^+ and x_t^- be the change in production (x_0^+, x_0^- : June to July).

Object function: $\min \sum_t 0.5x_t^+ + 0.25x_t^-$

Constraints: ① $4 + \sum_{i=0}^{t-1}(x_i^+ - x_i^- + S_{i+1}) = d_{i+1} \quad t=1,\cdots,12$

② $0 \leq 2 + \sum_{i=1}^{t} S_i \leq 10 \quad t=1,\cdots,12$

③ $x_i^+, x_i^-, d_i, S_i \geq 0$

①: Sales demands must be met. d_i is the sales demands of month i (d_1 for July).

②: Cannot exceed or overdraw from storage. S_i is the inventory used of month i (S_1 for July).

2.

Exercise 2.1 For each one of the following sets, determine whether it is a polyhedron.

(a) The set of all $(x, y) \in \Re^2$ satisfying the constraints

$$\begin{aligned} x\cos\theta + y\sin\theta &\leq 1, \qquad \forall\, \theta \in [0, \pi/2], \\ x &\geq 0, \\ y &\geq 0. \end{aligned}$$

(b) The set of all $x \in \Re$ satisfying the constraint $x^2 - 8x + 15 \leq 0$.

(c) The empty set.

Exercise 2.1 (a) Consider the polar coordinate system. Let $x = r\cos t$, $y = r\sin t$ $r \geq 0$, $t \in [0, 2\pi]$. Then $x\cos\theta + y\sin\theta \leq 1 \Leftrightarrow r\cos(\theta - t) \leq 1$ and $x \geq 0, y \geq 0 \Leftrightarrow t \in [0, \frac{\pi}{2}]$. Since the inequality must hold for all $\theta \in [0, \frac{\pi}{2}]$, we have $r \leq 1$. Therefore, the set actually is a quarter of a unit circle and hence is not a polyhedron.

(b)

$$x^2 - 8x + 15 \leq 0 \quad \Leftrightarrow \quad \begin{cases} x \leq 5 \\ x \geq 3 \end{cases}$$

Thus, the set is a polyhedron of the form $\{x \in \mathbb{R} | x \geq 3,\ x \leq 5\}$.

(c) Empty set is a polyhedron. An example is $\{x \in \mathbb{R} | x \geq 1,\ x \leq 0\}$.

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3. **Exercise 2.4** We know that every linear programming problem can be converted to an equivalent problem in standard form. We also know that nonempty polyhedra in standard form have at least one extreme point. We are then tempted to conclude that every nonempty polyhedron has at least one extreme point. Explain what is wrong with this argument.

Counter example: $\{(x,y)\in\mathbb{R}^2 \mid x+y\geq 1\}$. It has no extreme point.
Reason: Polyhedra in general form is different from polyhedra in standard form.