LECTURE 10: SENSITIVITY ANALYSIS

Today: What happens to our LP problem if we change one of our parameters (constraint, objective function, ...) just a little bit.

1. Shadow Price

Definition:

The **shadow price** of a constraint $ax \leq b$ is the change in the optimal solution z if we increase b by one unit.

Example: If we change a constraint from $2x_1 + 3x_2 \leq 5$ to $2x_1 + 3x_2 \leq 6$ and the optimal $z-$value changes from $z = 8$ to $z = 10$, then the shadow price of that constraint is $10 - 8 = 2$

Note: Here the change from 5 to 6 seems huge, but in practice more like a tiny change from 5000 to 5001, so the shadow price really measures the rate of change in the $z-$value, like a derivative.

Why do we care?

First of all, in practice, the constraints are not rigid, but flexible. Say a company says that their budget is \$5000. Then they can always change that number later, by negotiating for a higher budget.

Second of all, there are always round-off errors. For instance, we never have that the budget is *exactly* \$5000, in practice it's more like \$4997.30

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Sensitivity analysis tells us what happens to the new max without having to re-do the LP problem, which is computationally quite expensive.

2. Example

Example 1:

Find the shadow price of each constraint of the following LP

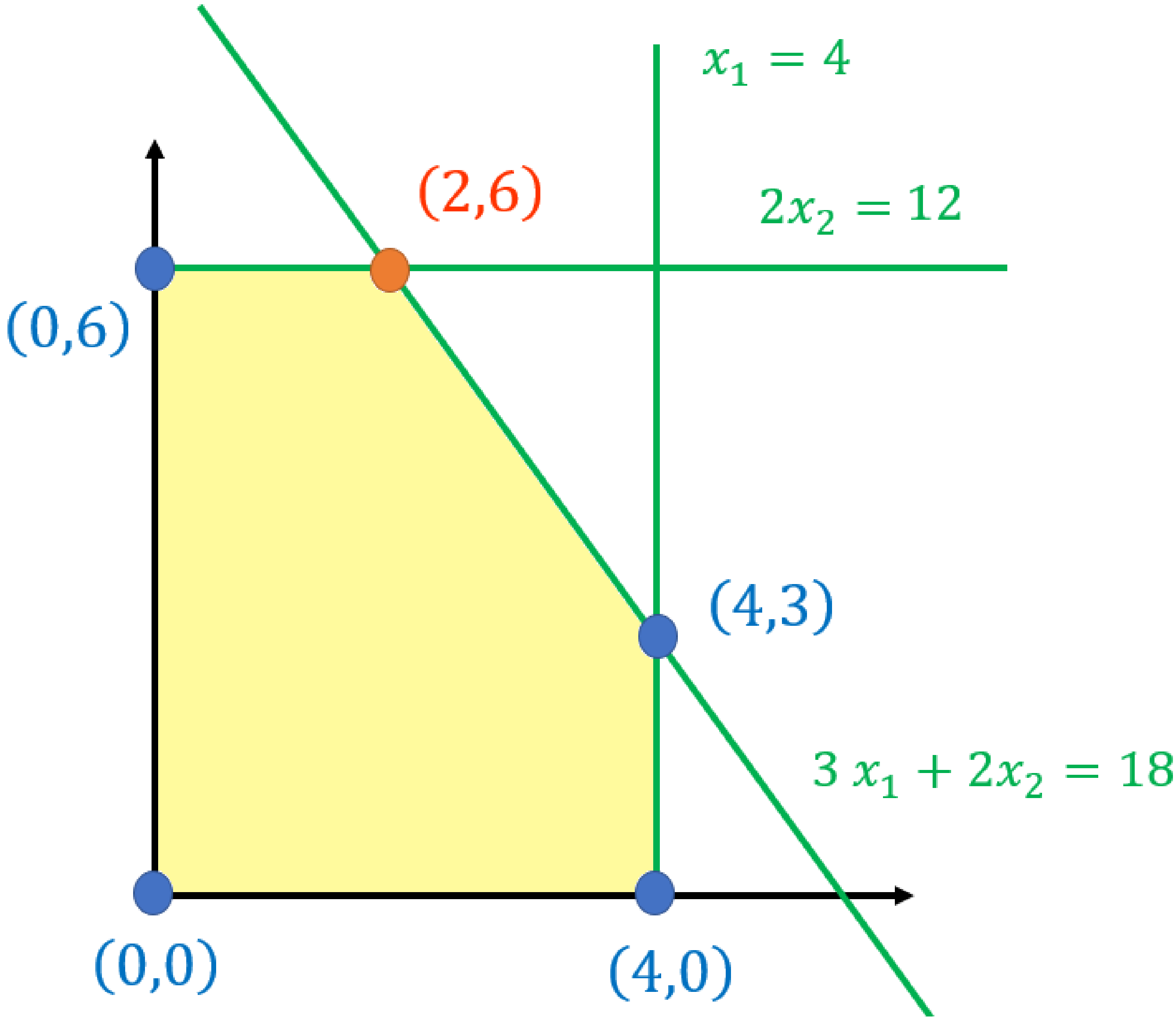
$$\begin{aligned} &\max 3x_1 + 5x_2 \\ \text{subject to } &x_1 \leq 4 && (1) \\ &2x_2 \leq 12 && (2) \\ &3x_1 + 2x_2 \leq 18 && (3) \\ &x_1 \geq 0 && (4) \\ &x_2 \geq 0 && (5) \end{aligned}$$

This is our model LP that we have solved a couple of times:

STEP 0: Original LP

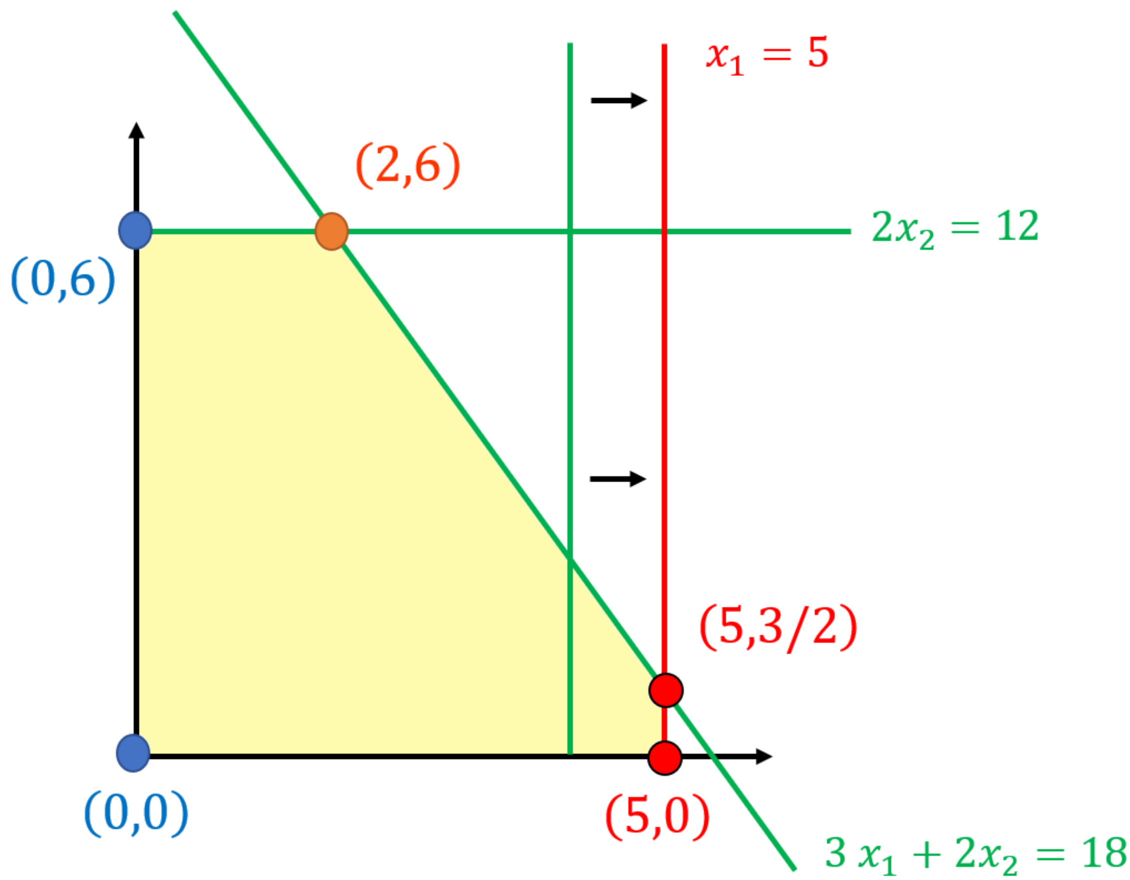
Optimal Vertex: $(2, 6)$

Optimal $z-$value: $z = 3(2) + 5(6) = 36$



STEP 1: Constraint ①

What if we do this problem but change $x_1 \leq 4$ to $x_1 \leq 5$



Then the only change is that the vertices $(4, 0)$ and $(4, 3)$ become $(5, 0)$ and $\left(5, \frac{3}{2}\right)$ (here we got $\frac{3}{2}$ by plugging in $x_1 = 5$ in $3x_1 + 2x_2 = 18$).

But $(5, 0)$ gives us $z = 15$ and $\left(5, \frac{3}{2}\right)$ gives us $z = 3(5) + 5\left(\frac{3}{2}\right) = 22.5$, neither of which is as good as $(2, 6)$

Optimal Vertex: $(2, 6)$

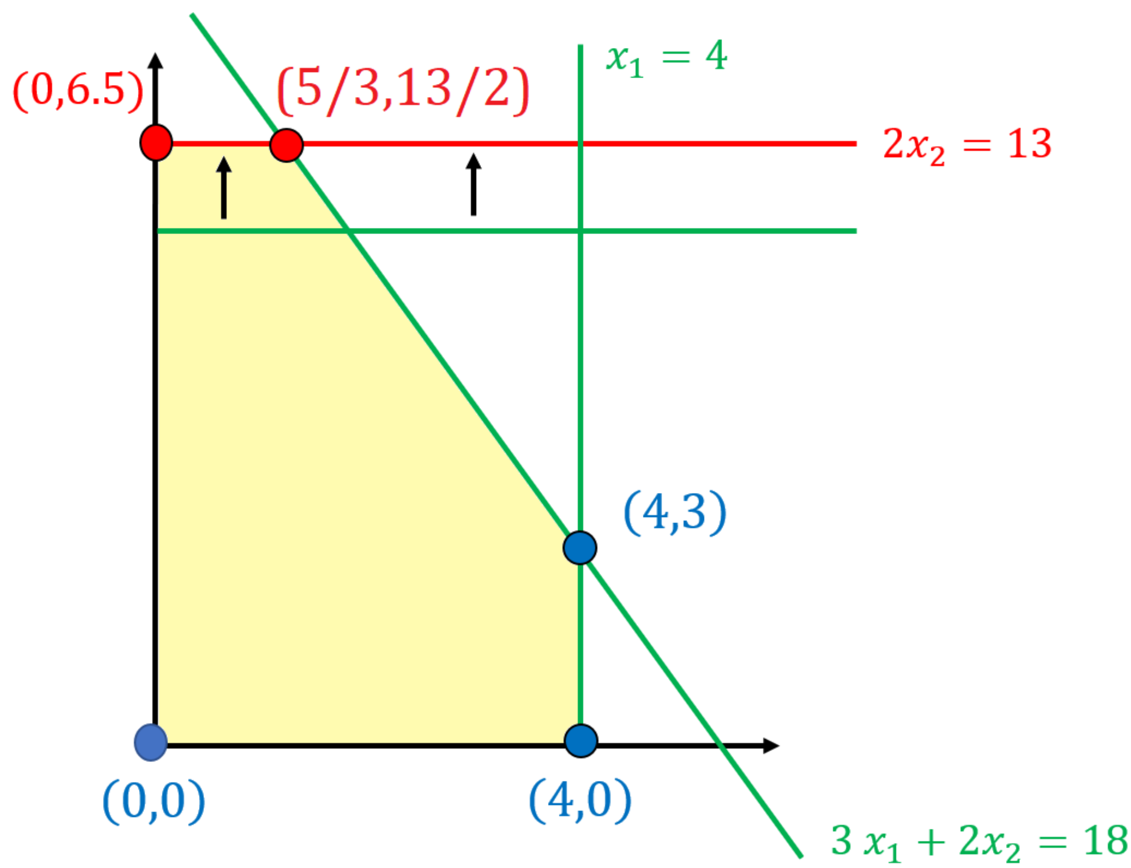
Optimal $z-$value: $z = 36$

Shadow price of ①: $p_1 = 36 - 36 = 0$

STEP 2: Constraint ②

Keep the **original** problem, but change $2x_2 \leq 12$ to $2x_2 \leq 13$

Note: It's important **NOT** to change $2x_2 \leq 12$ to $x_2 \leq 6$, otherwise you have a wrong shadow price



Here the vertices $(0,6)$ and $(2,6)$ become $\left(0,\frac{13}{2}\right)$ and $\left(\frac{5}{3},\frac{13}{2}\right)$ (we got $\frac{5}{3}$ by plugging in $x_2 = \frac{13}{2}$ in $3x_1 + 2x_2 = 18$)

For the optimal value, $\left(0, \frac{13}{2}\right)$ gives us $z = 0 + 5\left(\frac{13}{2}\right) = \frac{65}{2} = 32.5$

$\left(\frac{5}{3}, \frac{13}{2}\right)$ gives us $z = 3\left(\frac{5}{3}\right) + 5\left(\frac{13}{2}\right) = 5 + \frac{65}{2} = 5 + 32.5 = 37.5$

Note: Since the optimal vertex $(2, 6)$ changed, you technically also need to calculate the values of z at the other vertices, but $(0, 0)$ gives you $z = 0$ and $(4, 0)$ gives $z = 12$ and $(4, 3)$ gives $z = 27$

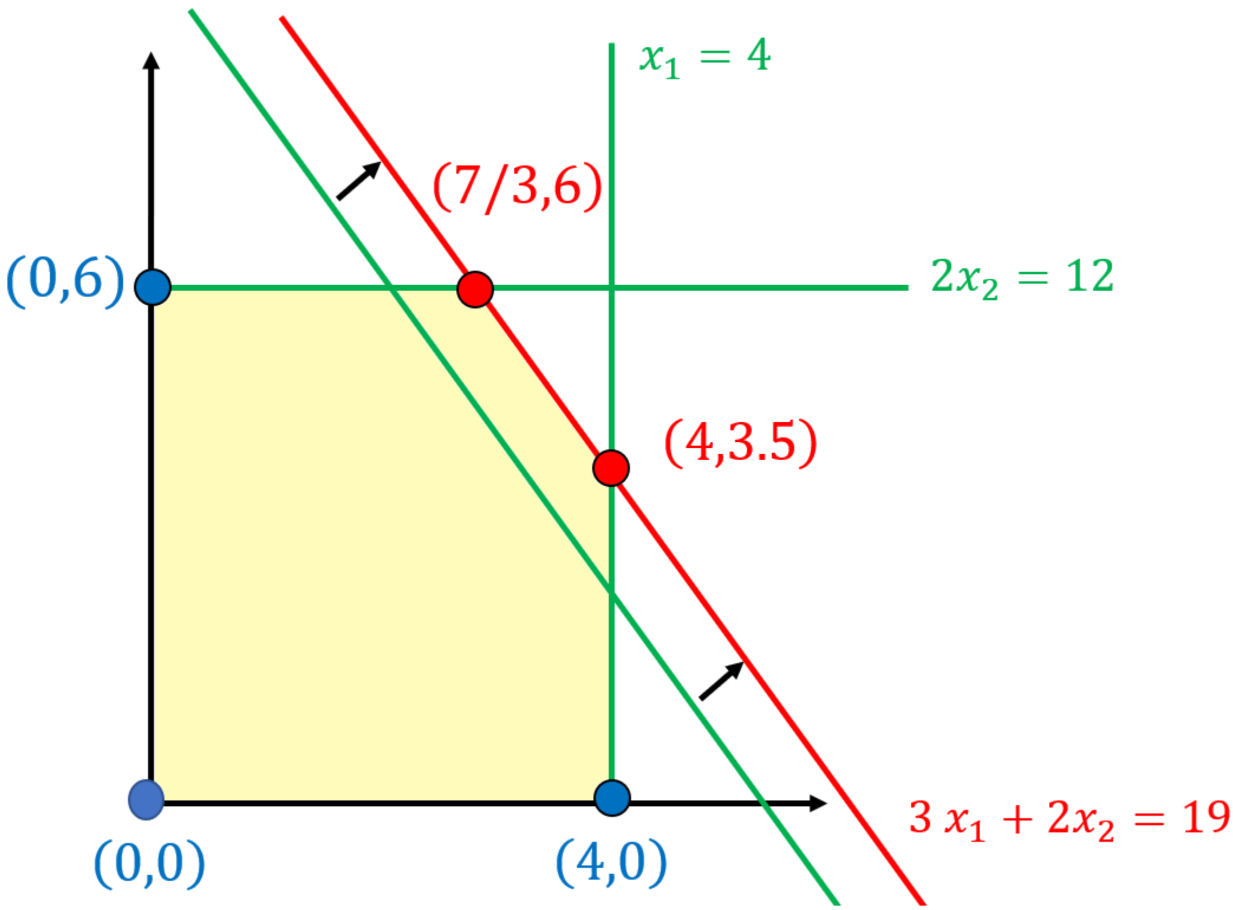
Optimal Vertex: $\left(\frac{5}{3}, \frac{13}{2}\right)$

Optimal $z-$value: $z = 37.5$

Shadow price of 2: $p_2 = 37.5 - 36 = 1.5$

STEP 3: Constraint 3

Change $3x_1 + 2x_2 = 18$ to $3x_1 + 2x_2 = 19$. The slope $-\frac{3}{2}$ is the same but the y-intercept is now $\frac{19}{2}$ instead of 9, so you just shift the line up.



Here the vertices $(4, 3)$ and $(2, 6)$ become $\left(4, \frac{7}{2}\right)$ and $\left(\frac{7}{3}, 6\right)$, which you get by setting $x_1 = 4$ in $3x_1 + 2x_2 = 19$ and $x_2 = 6$ in $3x_1 + 2x_2 = 19$.

For the optimal value, $\left(4, \frac{7}{2}\right)$ gives $z = 3(4) + 5\left(\frac{7}{2}\right) = 12 + \frac{35}{2} = 12 + 17.5 = 29.5$

$\left(\frac{7}{3}, 6\right)$ gives $z = 3\left(\frac{7}{3}\right) + 5(6) = 7 + 30 = 37$

And you can check that the other vertices are not optimal.

Optimal Vertex: $\left(\frac{7}{3}, 6\right)$

Optimal $z-$value: $z = 37$

Shadow Price of 3: $p_3 = 37 - 36 = 1$

Note: The shadow price could be negative, when the new z-value is smaller than the original one.

STEP 4: Constraint 4

Change $x_1 \geq 0$ to $x_1 \geq 1$

Similar to the above, you can check that this doesn't change optimal vertex and the optimal $z-$value

Shadow Price of 4: $p_4 = 0$

STEP 5: Constraint 5

Change $x_2 \geq 0$ to $x_2 \geq 1$

Similar to the above, you can check that this doesn't change optimal vertex and the optimal $z-$value

Shadow Price of 5: $p_5 = 0$

3. Relation to the Dual Problem

Keep in mind here that we got $p_1 = 0, p_2 = 1.5, p_3 = 1$. Surprisingly those shadow prices are related to the dual problem!

Primal LP:

$$\begin{aligned} \max\ & 3x_1 + 5x_2 \\ \text{subject to } & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Dual LP:

$$\begin{aligned} \min\ & 4y_1 + 12y_2 + 18y_3 \\ \text{subject to } & y_1 + 3y_3 \geq 3 \\ & 2y_2 + 2y_3 \geq 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

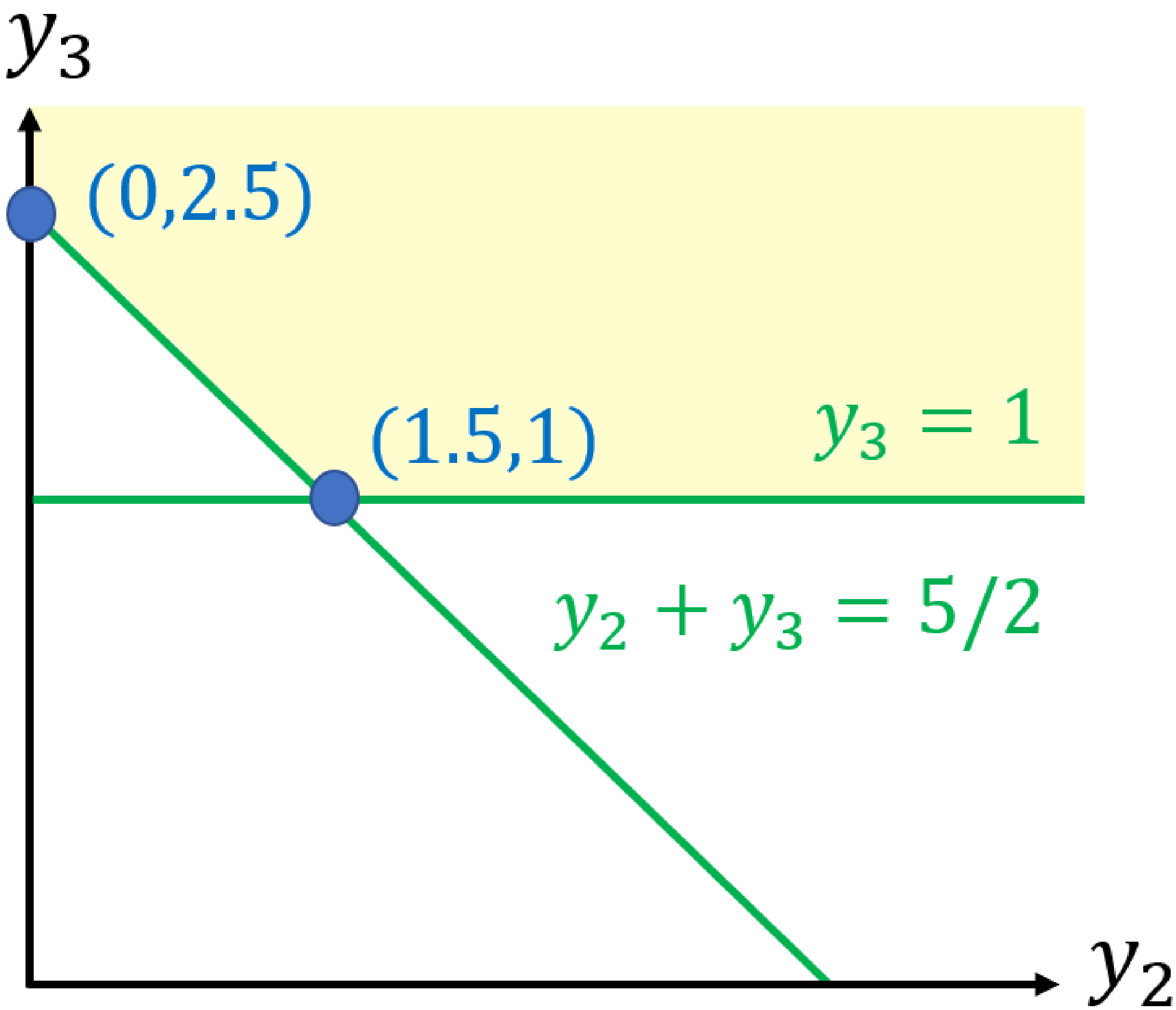
Recall: Using slack variables, we actually found that $y_1 = 0$ (since $s_1 = 2 > 0$)

This simplifies the problem to

Dual LP:

$$\begin{aligned} \min\ & 12y_2 + 18y_3 \\ \text{subject to } & 3y_3 \geq 3 \\ & 2y_2 + 2y_3 \geq 5 \\ & y_2, y_3 \geq 0 \end{aligned}$$

This is not particularly hard to solve: Since y_3 is more costly, we set $3y_3 = 3$ so $y_3 = 1$ and then $2y_2 + 2y_3 = 5$ becomes $2y_2 + 2 = 5$ so $y_2 = \frac{3}{2}$



(You can confirm this with vertices: Here there are two vertices, $\left(0, \frac{5}{2}\right)$ and $\left(1, \frac{3}{2}\right)$ and the second one gives you a smaller value. Here the feasible region is unbounded but it's ok here because z goes to ∞ in that region and we want the min value)

Miracle: Here the optimal dual variables $y_1 = 0, y_2 = \frac{3}{2}, y_3 = 1$ are the **same** as the shadow prices $p_1 = 0, p_2 = \frac{3}{2}, p_3 = 1$!! This is always true:

Fact:

The shadow prices p_j of the original LP are given by the values y_j of the optimal vertex of the dual LP

Which illustrates once again the usefulness of the dual problem! So in theory, if you solve the dual, you never have to calculate the shadow prices directly ever again!

4. Reduced Cost

Notice: In the previous example, changing the last two constraints $x_1 \geq 0$ or $x_2 \geq 0$, doesn't change the optimal $z-$value

This is because there was no optimal vertex of the form $(0, \star)$ or $(\star, 0)$.

If it were though, that would be kind of problematic.

Example: Suppose you're a farmer and your optimal vertex is $(0, 3)$. This tells you not to produce any wheat at all, so all the wheat field would go to waste!

Example: Remember the Peyamazon example from the beginning of the course where we saw that it's best not to give any free electronics to the customer!

And really, the issue is that sometimes the price of an object is too low. In the above example, the 3 in $3x_1 + 5x_2$ was too low, which made us favor x_2 as opposed to x_1

Question: If $x_j = 0$, can we raise the price of c_j of x_j to make $x_j > 0$?

(Or, in a minimization problem, can we *reduce the cost* of an object x_j to make $x_j > 0$?)

Definition:

The *reduced cost* $\overline{c_j}$ of $x_j = 0$ is the amount that the price c_j needs to increase so that that $x_j > 0$

And it turns out that, once again, this is related to the dual variables!

Fact:

To find $\overline{c_j}$, make the $j-$th dual constraint ≥ 0

(In terms of formulas, $\overline{c_j} = \sum_i a_{ij} y_i - c_j$, but it's easier to do it as below)

Example: Let's figure out $\overline{c_1}$ and $\overline{c_2}$ in the previous problem. Here $c_1 = 3$ (price of x_1) and $c_2 = 5$ (price of x_2)

Dual LP:

$$\begin{aligned} \min 4y_1 + 12y_2 + 18y_3 \\ \text{subject to } y_1 + 3y_3 \geq 3 \\ 2y_2 + 2y_3 \geq 5 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

And we found $y = \left(0, \frac{3}{2}, 1\right)$

$$\begin{aligned}\overline{c_1} =& y_1 + 3y_3 - 3 = 0 + 3(1) - 3 = 0 \\ \overline{c_2} =& 2y_2 + 2y_3 - 5 = 2\left(\frac{3}{2}\right) + 2(1) - 5 = 0\end{aligned}$$

So here we don't need to increase the prices of x_1 and x_2 at all. Makes sense because $x_1 = 0$ or $x_2 = 0$ are not optimal.

But if you found that $\overline{c_1} = 4$, then this says that you should increase the price of x_1 by 4 to actually sell some of it.

Note: For min problems, the reduced cost is the amount you need to decrease/reduce the cost c_j to get $x_j > 0$. In that case you make the jth dual constraint ≤ 0. And if you find $\overline{c_1} = -4$ then you would *reduce* the cost of x_1 by 4

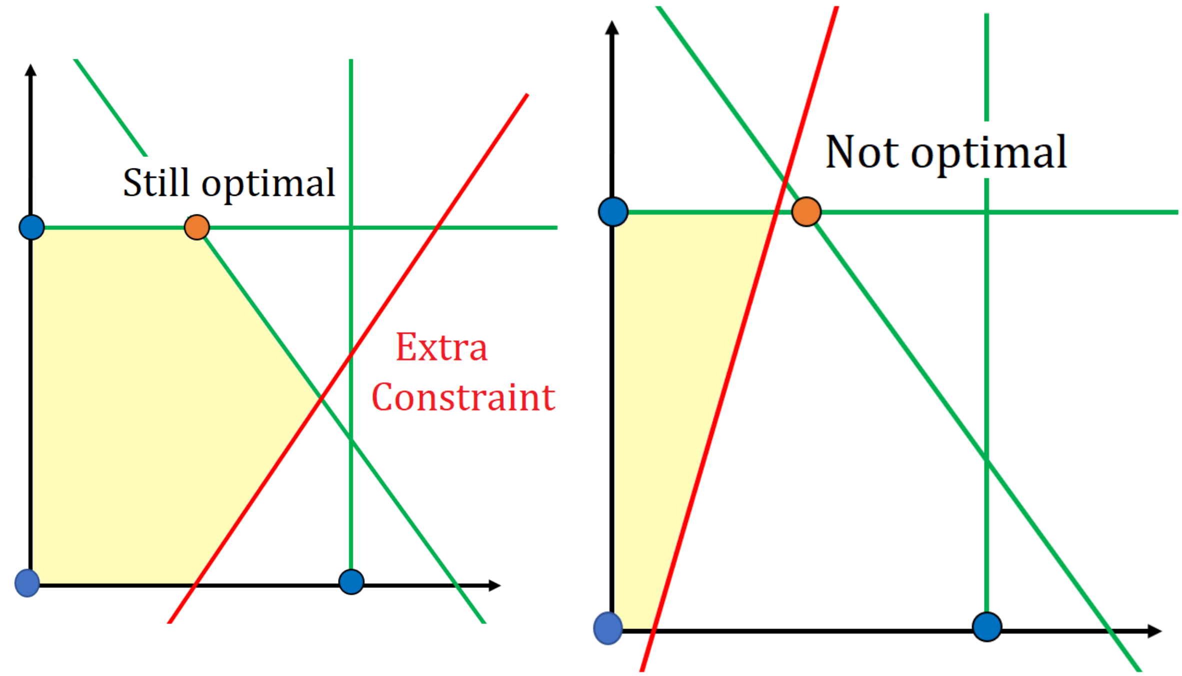
5. Other sensitivity issues

There are other things that we can change, other than the constraint values. If you're interested, the details are in the textbook.

Adding Constraints:

What if you add an extra constraint? Then you may or may not change the optimal vertex. But if you do, do you really need to start over the simplex method from scratch?

Turns out no, there are quicker methods in this case. In those methods, you use the old optimal to get partial information for the new



optimal, which speeds up the process.

Adding Variables:

What if you add a new variable? In the context of our example, say you add x_3. Then our original optimal vertex is $(2,6)$, and in some cases, it *could* happen that $(2,6,0)$ is a vertex of the new problem, and this one could be used as a starting vertex for the simplex method of the new problem. So if we're lucky, we don't even have to bother finding a starting vertex.

Keeping the optimal vertex:

If you change $z = 3x_1 + 5x_2$ to $z = 2.9x_1 + 5x_2$, that change of price is likely is not going to change the optimal vertex $(2,6)$, but if you change it too much, say $z = x_1 + 10x_2$, it's likely going to change the optimal vertex. So a related question is: "How much can we change the prices so as to not affect the optimal vertex?"

And similarly for the constraints, how much can we change the constraints so as to not to affect the $(2, 6)$. Think of it in terms of a stubborn company that really doesn't want to change its means of production and wants to stay at $(2, 6)$.