LECTURE 13: NETWORK PROBLEMS (I)

Today: Graph Theory and its relation to LP problems.

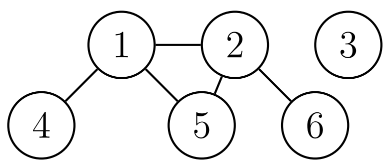
1. Basic Lingo

Definition:

A **graph** $G = (V, E)$ is just a set of points V called **vertices** (or nodes) connected by **edges** E

Example 1:

Undirected Graph



Here the vertices are

$$V = \{1, 2, 3, 4, 5, 6\}$$

And the edges are

$$E = \{\{1, 2\}, \{1, 4\}, \{1, 5\}, \{2, 5\}, \{2, 6\}\}$$

Means that 1 and 2 are connected, 1 and 4 are connected etc.

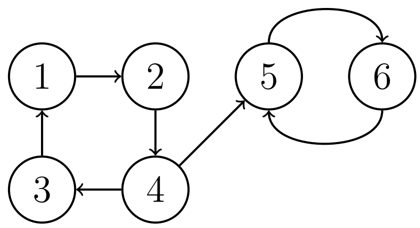
Date: Thursday, October 20, 2022.

This is an **undirected graph** in the sense that we can go from 1 to 2 and back from 2 to 1.

Sometimes can have **directed graphs:**

Example 2:

Directed Graph



Means that you can go from 1 to 2, but not from 2 to 1

Think for example in video games where you enter a room, but then the door closes behind you, so you can't go back

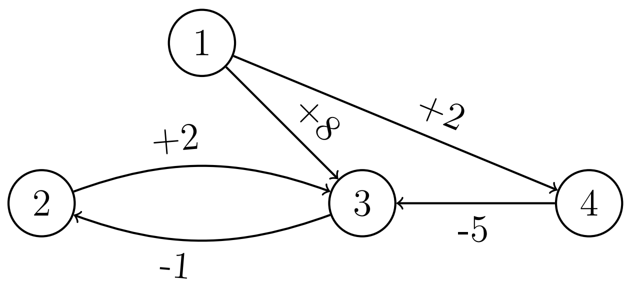
$$V = \{1, 2, 3, 4, 5, 6\}$$
$$E = \{(1, 2), (2, 4), (4, 3), (3, 1), (4, 5), (5, 6), (6, 5)\}$$

The parentheses are used to emphasize that $(1, 2) \neq (2, 1)$

Finally, we can assign values on the edges, think like toll roads, or costs of plane tickets between cities

Example 3:

Values at edges



Going from 1 to 4 would give you $2 but going from 4 to 3 would cost you $5

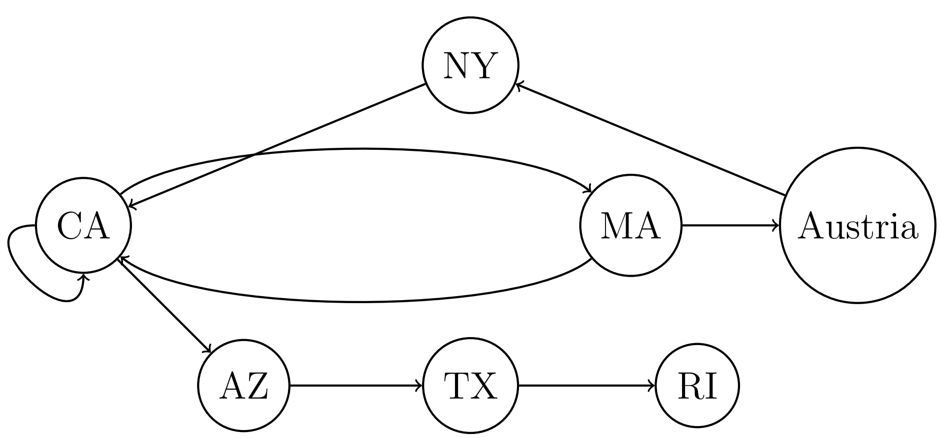
2. Examples and Applications

Graphs/Networks appear everywhere in life and there is no escaping them ☺

Here is a graph of where I'm from

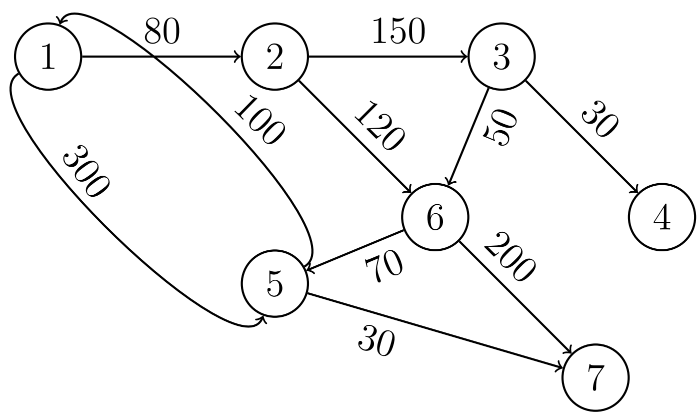
Example 4:

Peyam Graph



Example 5: (Flight Prices)

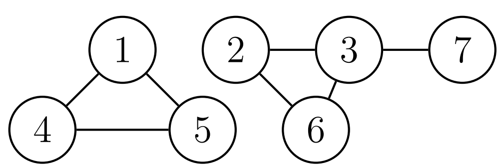
Flight Prices



Here the vertices are cities and the edges are flight prices. Then an important question is to find the **optimal path** from city 1 to city 7, the one that would be the least expensive one.

Example 6:

Social Networks

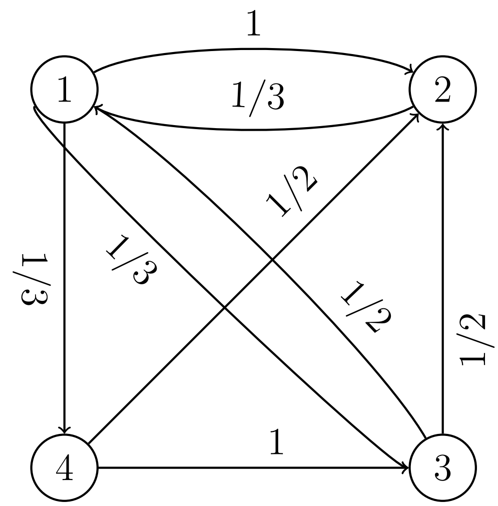


Here the vertices are people, and an edge means "X is friends with Y." Very important for example in determining who has been in close contact with someone who got COVID.

Note: At some point there was an experiment called "Six Degrees of Separation," where basically the question was "Can you go from one random person to another one by using only 6 edges?"

Example 7:

Google/PageRank



Here the vertices are 4 websites, and we link two websites if one references the other one. Here for example, website 1 references 2, 3, 4. The numbers represent probabilities/likelihoods. For example, the probability of going from website 1 to website 4 (in one step) is $\frac{1}{3}$, since there are 3 edges going out of 1.

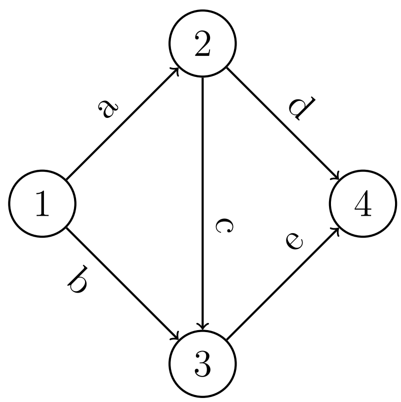
The idea behind the PageRank algorithm is: If you start with a vertex at random and roll a die and choose an edge according to the probabilities above, and do this infinitely many times, where are you most likely going to end up? That would be your highest ranked website, followed by the second highest one etc.

3. Adjacency and Incidence Matrix

We can conveniently represent graphs using matrices

Example 8:

Find the **adjacency matrix** A and the **oriented incidence matrix** M of the following graph



Here A is a 4×4 matrix ($4 =$ number of vertices)

Definition:

$$a_{ij} = \begin{cases} 1 & \text{if vertices } i \text{ and } j \text{ are connected by an edge} \\ 0 & \text{otherwise} \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And M is a 4×5 matrix ($4 =$ number of vertices, $5 =$ number of edges)

Definition:

$$m_{ij} = \begin{cases} 1 & \text{if edge } j \text{ points out from vertex } i \\ -1 & \text{if edge } j \text{ points in from vertex } i \\ 0 & \text{otherwise} \end{cases}$$

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

For example, for the second row, look at vertex 2. Edge a points towards 2 (so -1), edge b is unrelated to 2 (so 0), edge c points away from 2 (so 1), same for edge d (so 1) and edge e is unrelated (so 0).

Note: Here the columns of M sum to 0. This is because a given edge going out from one point (1) has to go into another point (-1)

4. General Network Flow Problem

Let's see how we can translate graph theory into an LP problem, via a concrete example:

Suppose you're a coffee producing company and

(1) Your coffee get produced in city 1 and shipped to city 4, via the intermediate cities 2 and 3. Here the **vertices** are the cities.

(2) Your coffee get transported to the cities through a network of roads. Here the **edges** represent the roads connecting the city.

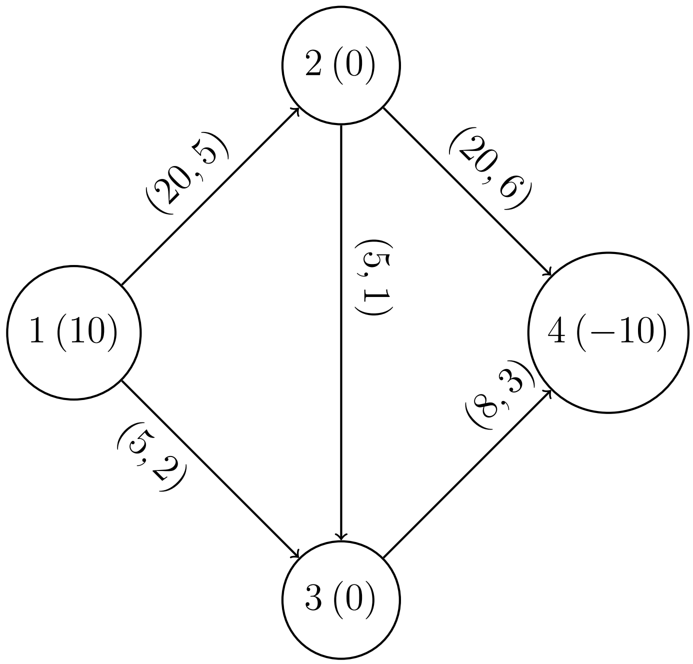
(3) Each road/edge has a **capacity constraint**, think weight limit (or with virtual network it might be a bandwidth limit)

(4) Each road/edge has a **shipping cost**

Goal: Route our items via the networks ② to meet the demands of cities ①, satisfying the constraints of ③ and minimizing the total cost given by ④

Example 9:

Suppose our graph looks like this:



Here is how to read this graph:

1(10) means vertex 1 with supply 10 lbs of coffee. Here is where the coffee is produced

2(0) and 3(0) mean vertices 2 and 3 with no coffee. Those are transit cities.

4(−10) means vertex 4 with *demand* 10 lbs of coffee. Here is where coffee needs to be sent.

$(20, 5)$ means that the maximum capacity of that road is 20 lbs and the cost/toll is \$5.

Problem: How can we route 10 lbs of coffee at a minimal cost?

There are several solutions, not all of them optimal. For example we can ship the 10 lbs of coffee via the route $(1,2) \to (2,4)$. Then the cost is

$$10 \times 5 + 10 \times 6 = 110$$

Or we can ship 5 lbs of coffee via the route $(1,2) \to (2,4)$, and the other 5 lbs of coffee via the routes $(1,3) \to (3,4)$, then the cost is

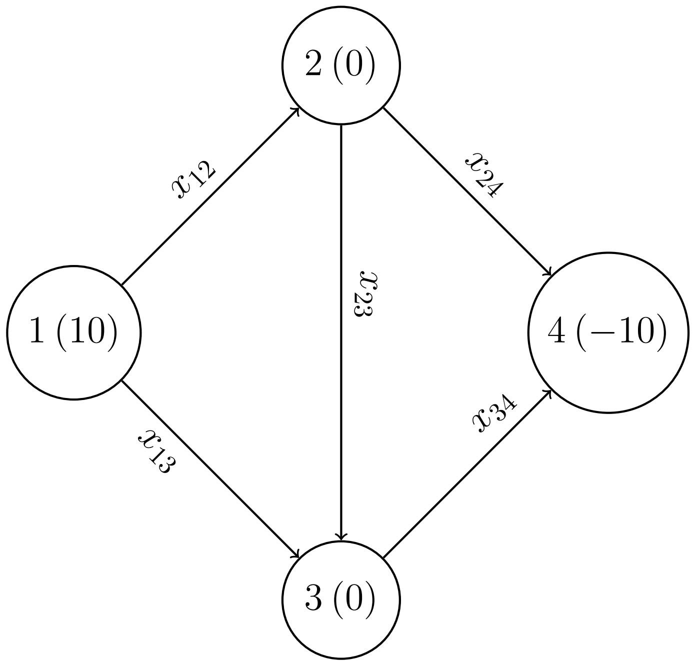
$$5 \times 5 + 5 \times 6 + 5 \times 2 + 5 \times 3 = 80$$

Which is much less expensive!

But we can't for example put all the 10 lbs on the route $(1,3)$ since it would exceed the capacity of 5 lbs.

5. LP Formulation

Decision Variables: x_{ij} = amount of coffee transported on the edge (i,j)



Constraints: There are two kinds of constraints:

Conservation-of-flow constraints:

First of all, the amount of coffee leaving city 1 has to be 10, so

$$x_{12} + x_{13} = 10$$

The amount of coffee going into city 2 = amount of coffee going out of it, so

$$x_{12} = x_{24} + x_{23} \Rightarrow -x_{12} + x_{23} + x_{24} = 0$$

Same for city 3

$$-x_{13} - x_{23} + x_{34} = 0$$

The amount of coffee going in city 4 is 10, so

$$x_{24} + x_{34} = 10 \Rightarrow -x_{24} - x_{34} = -10$$

Capacity constraints:

$$\begin{aligned} x_{12} &\leq 20 \\ x_{13} &\leq 5 \\ x_{23} &\leq 5 \\ x_{24} &\leq 20 \\ x_{34} &\leq 8 \\ x_{ij} &\geq 0 \end{aligned}$$

Objective Function:

$$z = 5x_{12} + 6x_{24} + 1x_{23} + 2x_{13} + 3x_{34}$$

General LP Problem:

$$\begin{aligned} \min z = {} & 5x_{12} + 2x_{13} + 1x_{23} + 6x_{24} + 3x_{34} \\ \text{subject to } & x_{12} + x_{13} = 10 \\ & -x_{12} + x_{23} + x_{24} = 0 \\ & -x_{23} - x_{13} + x_{34} = 0 \\ & -x_{24} - x_{34} = -10 \\ & x_{12} \leq 20 \\ & x_{13} \leq 5 \\ & x_{23} \leq 5 \\ & x_{24} \leq 20 \\ & x_{34} \leq 8 \\ & x_{ij} \geq 0 \end{aligned}$$

Note: If you write this in matrix form, the incidence matrix M shows up, which makes sense given the way we set up the problem!