LECTURE 4: GEOMETRY OF FEASIBLE REGIONS + CONVEXITY

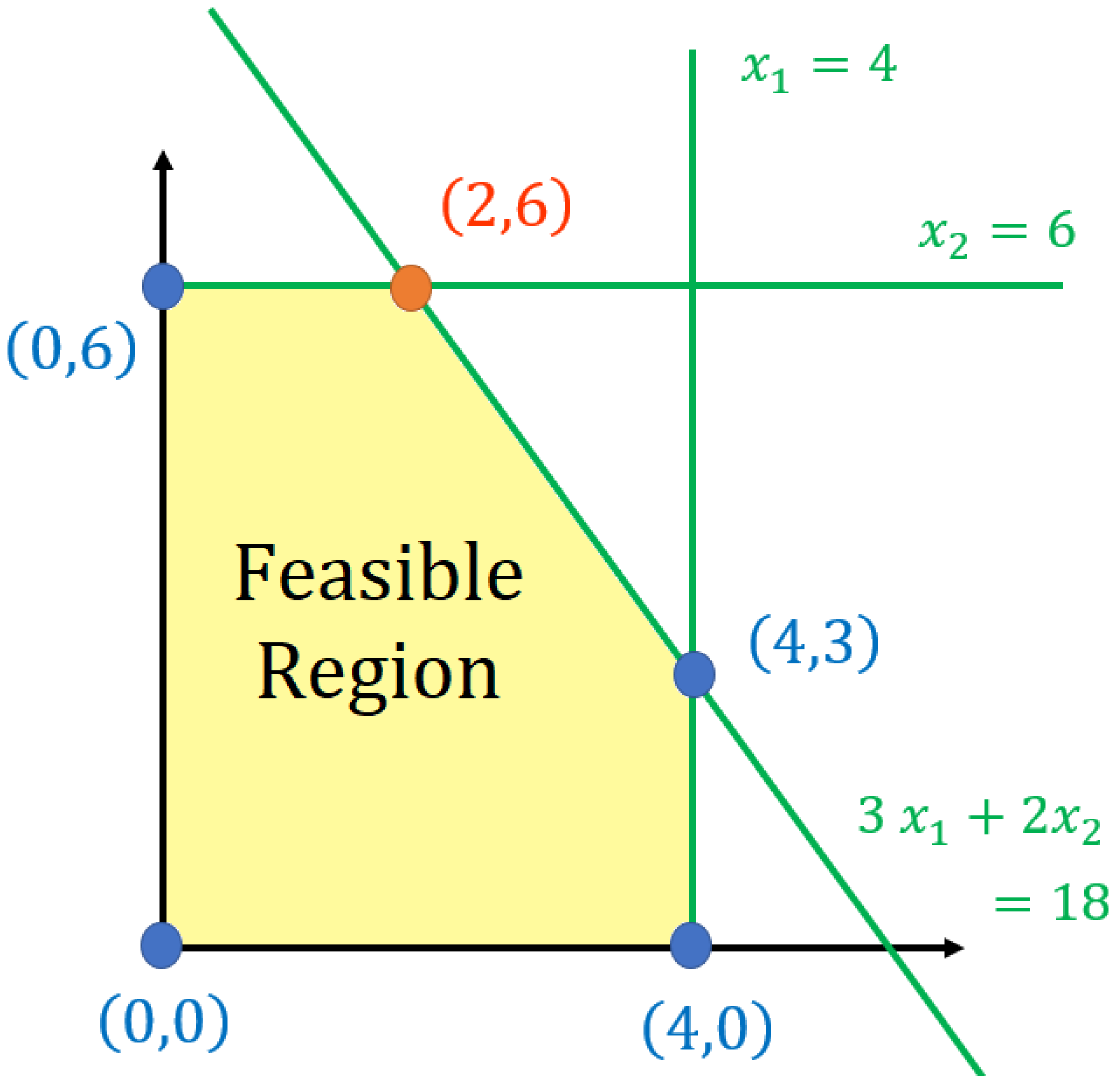
1. Motivation

Last time: Solved the following linear programming problem:

Example

$$\begin{aligned} \max z =& 3x_1 + 5x_2 \\ \text{subject to } & x_1 \leq 4 \\ & x_2 \leq 6 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Date: Tuesday, September 20, 2022.



We saw that the optimal solution is attained at one of the vertices (corner points) of the feasible region.

Because of this, we will soon devise an algorithm called the **simplex algorithm** that will go through the vertices of the feasible region and figure out which one is optimal.

Question: Is there an *algebraic* way to describe the corner points? Because even though it is obvious to us what those are, it's not as obvious for a computer. And it's even less obvious if the feasible region is three-dimensional (think a soccer ball) or even four-dimensional!

This is why today is mostly a geometry lesson about feasible regions.

2. Hyperplanes and Polyhedra

Definition: A **polyhedron** is the set of points of the form

$$\{x \in \mathbb{R}^n \mid Ax \leq b\}$$

Where A is a $m \times n$ matrix and b is a vector in $\mathbb{R}^m$.

Notice our feasible region is precisely of this form.

Each individual constraint like $3x_1 + 2x_2 \leq 18$, has its own particular significance. Notice that you can write this one as $a^T x \leq b$ where $a = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $b = 18$

Definition: A **hyperplane** is the set of points of the form

$$\left\{x \in \mathbb{R}^n \mid a^T x = b\right\}$$

Where a is a nonzero vector and b is a scalar.

Compare this with $ax + by + cz = d$ from multivariable calculus.

Fact: a is always perpendicular to the hyperplane $a^T x = b$

Why? Fix two vectors x and y in that hyperplane and consider $z = y - z$ (see picture in lecture) then

$$a \cdot z = a^T z = a^T (y - x) = a^T y - a^T x = b - b = 0$$

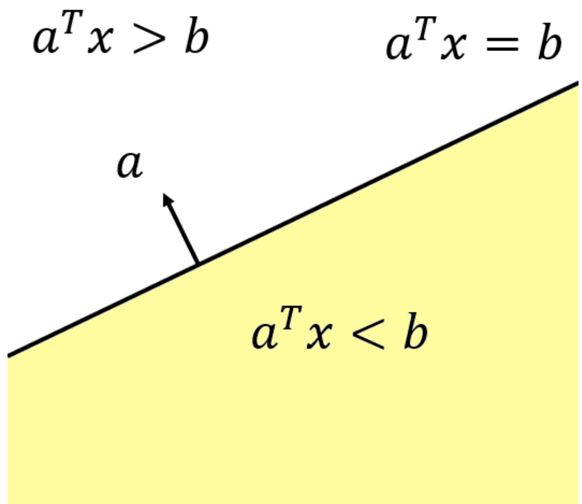
Hence a is perpendicular to z ✓

Now if we have an inequality, then it's called a half-space:

Definition: A **half-space** is the set of points of the form

$$\left\{ x \in \mathbb{R}^n \mid a^T x \leq b \right\}$$

It is precisely one side of the hyperplane $a^T x = b$



Note: Each constraint represents a half-space, and so in the end, the feasible region is just an intersection of half-spaces.

Fact: A polyhedron is the intersection of a finite number of half-spaces

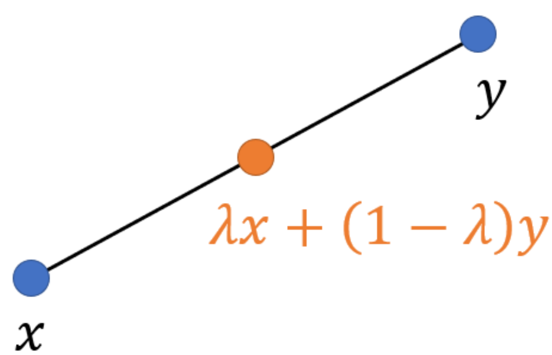
(This is because each row of $Ax \leq \mathbf{b}$ is just an equation of the form $ax \leq b$, where $a = i^{th}$ row of A and $b = i^{th}$ entry of $\mathbf{b}$)

3. Convex Sets

The feasible regions we talk about have a special structure called **convexity**, which plays an important role in this course (and applied math in general)

Recall: The line segment between x and y can be represented by

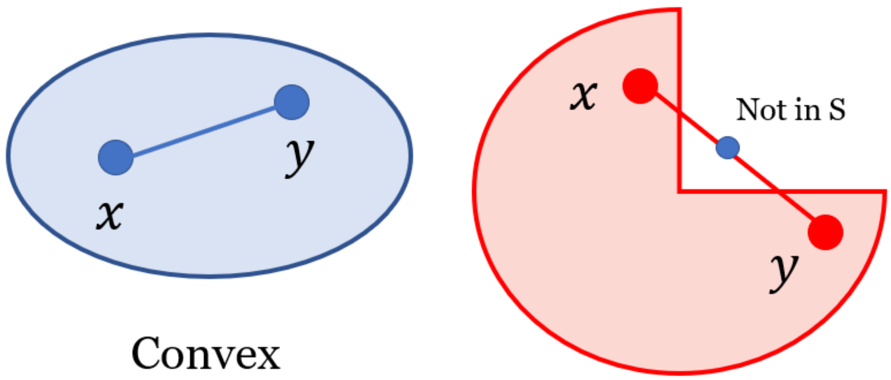
$$\begin{cases} \lambda x + (1-\lambda) y \\ 0 \leq \lambda \leq 1 \end{cases}$$



For example, $\lambda = 0$ corresponds to y, $\lambda = 1$ corresponds to x and $\lambda = \frac{1}{2}$ corresponds to the midpoint between x and y

Convex just means that for every pair of points x and y on the region, the line segment between x and y is also in the region.

Definition: A set S is **convex** if for any $x, y \in S$, and any $0 \leq \lambda \leq 1$ we have $\lambda x + (1-\lambda)y \in S$



Convex Not Convex

Fact: The intersection of convex sets is convex

(See book, just an application of the definition of convexity)

As mentioned above, our feasible regions will always be convex:

Fact: Every polyhedron is a convex set.

Why? It's enough to show that half-spaces $a^T x \leq b$ are convex, because then we can use the fact that a polyhedron is the intersection of half-spaces and the intersection of convex sets is convex.

Suppose x and y satisfy $a^T x \leq b$ and $a^T y \leq b$, then

$$a^T(\lambda x + (1-\lambda)y) = \lambda a^T x + (1-\lambda)a^T y \leq \lambda b + (1-\lambda)b = b$$

Hence $\lambda x + (1-\lambda)y$ is in the half-space as well $\square$

Convex Combinations: It's useful to generalize the quantity $\lambda x + (1-\lambda)y$ to the case of three or more points:

Definition: A **convex combination** of vectors $x_1, x_2, \ldots, x_k$ is an expression of the form

$$\lambda_1 x_1 + \cdots + \lambda_k x_k$$

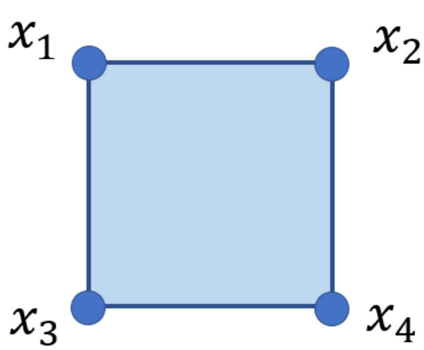
Where $\lambda_1 + \cdots + \lambda_k = 1$ (and the λ_i are non-negative)

Note: In the case $k = 2$ we get $\lambda_2 = 1 - \lambda_1$ and the above expression becomes $\lambda_1 x_1 + (1 - \lambda_1)x_2$

Definition: The **convex hull** of $x_1, \ldots, x_k$ is the set of all convex combinations of those vectors.

It's actually the smallest convex set containing $x_1, \ldots, x_k$

Example: The convex hull of the four points $x_1, \ldots, x_4$ in the figure is a square. It contains the points, as well as the line segments between them, but also other convex combinations as well, like $\frac{1}{4}x_1 + \frac{1}{2}x_3 + \frac{1}{4}x_4$



Why this matters? So far, we defined a polyhedron as being the intersection of half-spaces, but usually you think of polytopes in terms of vertices, and in fact a more natural definition is the following

(Alternative) Definition: A **polyhedron** is the convex hull of finitely many points

The fact that both definitions are equivalent is not trivial at all, and the proof will be skipped. Instead, let's illustrate with an example:

Example: Consider the unit cube in $\mathbb{R}^3$. One way is to define it was the convex hull of $(1,0,0), (0,1,0)$, etc.

Another way to define it is as the intersection of the 6 half-spaces

$$\begin{array}{ll} -x_1 \leq 0 & x_1 \leq 1 \\ -x_2 \leq 0 & x_2 \leq 1 \\ -x_3 \leq 0 & x_3 \leq 1 \end{array}$$

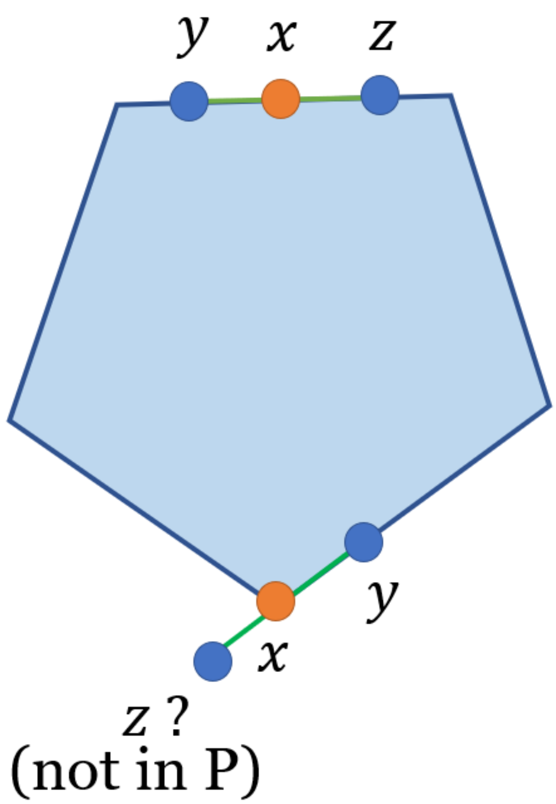
Both formulations have their advantages and disadvantages. The half-space definition is more algebraic, while the convex hull definition focuses on the points, which could or could not be vertices

4. Vertices

And in fact, we are finally ready to define what a vertex is. We will give 3 definitions, which turn out to be equivalent.

The first one has to do with **extreme points**.

Intuitively, if a point is **not** a vertex, then it must be on a segment between two points, like in this figure:



Let P be a polyhedron

Definition: $x \in P$ is **not an extreme point** of P if there are $y, z \in P$ with $y \neq x, z \neq x$ and $0 \leq \lambda \leq 1$ such that

$$x = \lambda y + (1 - \lambda) z$$

Definition 1: [Extreme Points]

$x \in P$ is an **extreme point** if $x = \lambda y + (1 - \lambda) z$ with y, z, λ as above implies that $x = y$ or $x = z$

An alternative definition of vertices is in terms of a minimization problem.

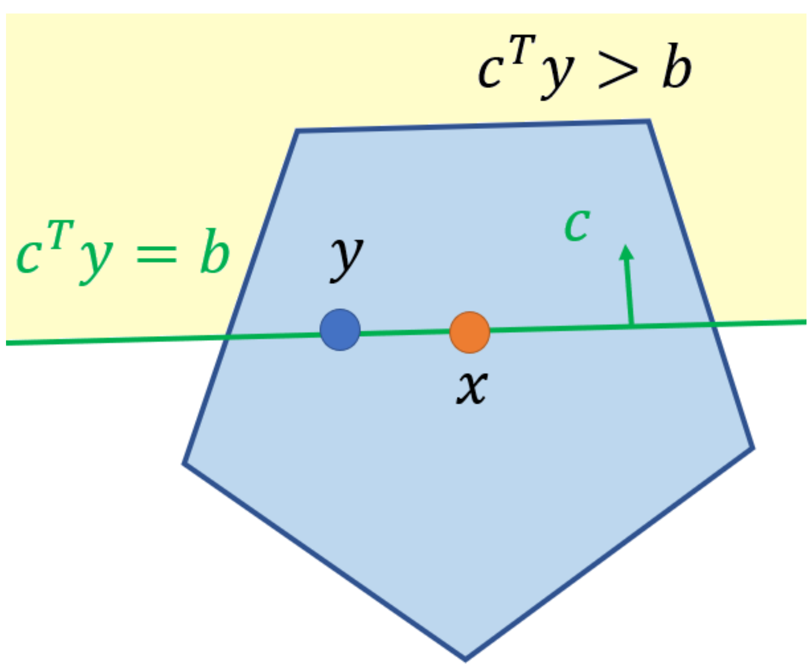
Definition 2: [Vertex] $x \in P$ is a **vertex** of P if there is c such that for all $y \neq x \in P$ we have

$$c^T x < c^T y$$

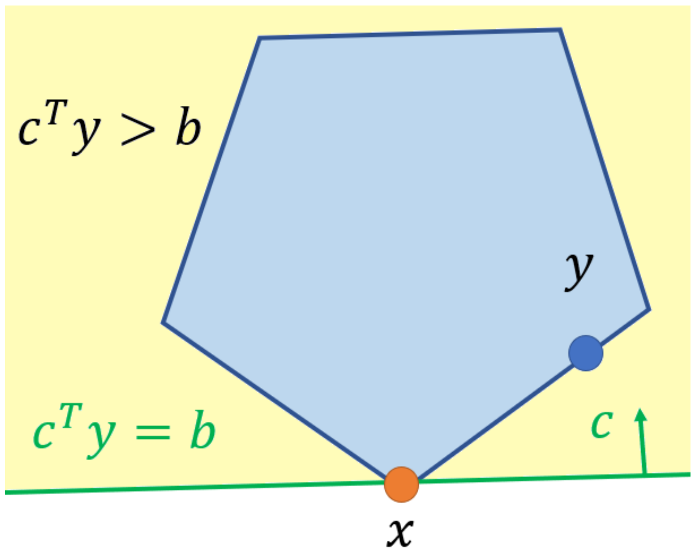
In other words, $c^T x$ is the strict minimum of the function above.

Geometric Interpretation: If you let $b = c^T x$ (fixed) then the above says that any point in P other than x lies in the half-space $\left\{c^T y > b\right\}$, so P lies entirely on one side of the half-plane $\left\{c^T y = b\right\}$, as in the figure below:

Not vertex:



Vertex:



(Think like balancing P on the tip of your finger)

While those definitions are nice from a geometric point of view, they are difficult to implement with an algorithm. What we need is a purely algebraic definition, which I'll explain through an example

Example:

$$\begin{aligned} \max z &= \text{Blah} \\ \text{subject to } x_1 + 2x_2 &\leq 5 \\ x_1 - 3x_2 &\leq 7 \\ x_2 + x_3 &= 4 \\ x_1 + 3x_3 &= 1 \end{aligned}$$

Notice some constraints come with $=$ and some come with $\leq$.

Also, *could* happen that some $\leq$ constraints are satisfied with equality, like it could happen that at a specific point (x_1, x_2, x_3), we have $x_1 + 2x_2 = 5$, we call this **active/binding** at (x_1, x_2, x_3).

Intuitively, a vertex (x_1, x_2, x_3) here just means that

(1) All the constraints with $=$ are satisfied (here the third and fourth one)

(2) There are a total of 3 (linearly independent) active constraints

(Precise definition next time)