LECTURE 8: LP DUALITY (I)

Today: Solve a LP problem by turning it into another LP problem!

1. Motivation

Example 1:

$$\begin{aligned} \max\ z = & x_1 + 6x_2 \\ \text{subject to } & x_1 \leq 20 \quad \text{①} \\ & x_2 \leq 30 \quad \text{②} \\ & x_1 + x_2 \leq 40 \quad \text{③} \\ & x_1, x_2 \geq 0 \end{aligned}$$

Let's play around with the constraints a little bit: How about we take

$$\text{①} + 6 \times \text{②}$$

$$\begin{aligned} \text{Then } \ (x_1) + 6\,(x_2) \leq & (20) + 6(30) \\ z \leq & 200 \end{aligned}$$

Therefore, z is <u>at most</u> 200

Another way of getting $x_1 + 6x_2$ is by using

$$5 \times \text{②} + \text{③}$$

$$\begin{aligned} 5\,(x_2) + (x_1 + x_2) \leq & 5(30) + (40) \\ x_1 + 6x_2 \leq & 190 \\ z \leq & 190 \end{aligned}$$

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So z is at most 190

Notice here that we want to **minimize** the right-hand-side, this will be important below.

Let's generalize this by taking

$$\begin{aligned} y_1 \times ① + y_2 \times ② + y_3 \times ③ \\ y_1(x_1) + y_2(x_2) + y_3(x_1 + x_2) \leq & y_1(20) + y_2(30) + y_3(40) \\ (y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq & 20y_1 + 30y_2 + 40y_3 \end{aligned}$$

Now **IF** $\boxed{y_1 + y_3 \geq 1}$ and $\boxed{y_2 + y_3 \geq 6}$ then we have

$$\begin{aligned} z = & 1x_1 + 6x_2 \\ \leq & (y_1 + y_3)x_1 + (y_2 + y_3)x_2 \\ \leq & 20y_1 + 30y_2 + 40y_3 \end{aligned}$$

z is at most $20y_1 + 30y_2 + 40y_3$, and our goal is to find y_1, y_2, y_3 to make this as small as possible.

Summary: We have turned our LP into the following problem

$$\begin{aligned} \min z = & 20y_1 + 30y_2 + 40y_3 \\ \text{subject to } & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 6 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

This called the **dual LP problem**

Note: The idea is that sometimes this is easier to solve than our original problem, especially with MATLAB, who loves min problems!

2. The Dual LP Problem

We can now give a formal definition

Original/Primal LP Problem:

$$\begin{aligned} &\max c^T x \\ &\text{subject to } Ax \leq b \\ &\qquad x \geq 0 \end{aligned}$$

Dual LP Problem:

$$\begin{aligned} &\min b^T y \\ &\text{subject to } A^T y \geq c \\ &\qquad y \geq 0 \end{aligned}$$

In the example above, we had $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ with $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $c = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ and $b = \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix}$, so this checks out. Carefully note how the prices $1, 6$ become the constraint, and the constraint $20, 30, 40$ becomes the new price.

Example 2:

Find the dual to

$$\begin{aligned} \max\ & 2x_1 + 3x_2 + 5x_3 \\ \text{subject to } & x_1 + 2x_2 + 4x_3 \leq 10 \\ & 5x_1 + x_2 + 3x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

3 variables and 2 constraints now become 2 variables and 3 constraints. Remember to swap the prices and the constraints to get:

$$\begin{aligned} \min\ & 10y_1 + 20y_2 \\ \text{subject to } & y_1 + 5y_2 \geq 2 \\ & 2y_1 + y_2 \geq 3 \\ & 4y_1 + 3y_2 \geq 5 \\ & y_1, y_2 \geq 0 \end{aligned}$$

(Here for each constraint we just used the columns of A, so for example the first column of A is $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ which gives $y_1 + 5y_2$)

Why dual? Think of dual as "related." We will soon see that the dual problem is related to the original problem. In practice, we're hoping that the dual problem is easier to solve.

Zelda-Analogy: If the primal problem is like a real world, then the dual problem is like a shadow world. It's kind of the same problem, but with a different perspective.

Sign Rules:

Example 3:

Find the dual to

$$\begin{aligned} \max\ & x_1 + 3x_2 + 5x_3 \\ \text{subject to } & 3x_1 + 4x_2 \leq 2 \\ & x_1 + 5x_2 \geq 4 \\ & 2x_1 + 7x_3 = 6 \\ & x_1 \geq 0 \\ & x_2 \leq 0 \\ & x_3 \text{ unconstrained} \end{aligned}$$

- The sign of y_j is **opposite** of the sign of the original constraint b_j (so here you flip the $\leq, \geq$, and $=$ from the constraints)
- The sign the new constraint c_i is **the same** as the sign of the original variable x_i (so here you keep the order $\geq, \leq$, and $=$ from the x_i)

$$\begin{aligned} \min\ & 2y_1 + 4y_2 + 6y_3 \\ \text{subject to}\ & y_1 \geq 0 \\ & y_2 \leq 0 \\ & y_3 \text{ unconstrained} \\ & 3y_1 + y_2 + 2y_3 \geq 1 \\ & 4y_1 + 5y_2 + 0y_3 \leq 3 \\ & 0y_1 + 0y_2 + 7y_3 = 5 \end{aligned}$$

(This is much easier, because notice how we can solve for y_3 already)

3. Weak Duality

Fact:

The dual of the dual is the primal

(Compare this to $\left(A^T\right)^T = A$ in linear algebra)

Why? The dual to

$$\begin{aligned} \min\ & b^T y \\ \text{subject to}\ & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

Becomes the following

$$\begin{aligned} \max\ & c^T x \\ \text{subject to}\ & \underbrace{\left(A^T\right)^T x}_{Ax} \leq b \\ & x \geq 0 \end{aligned}$$

Which is the primal problem □

The next fact gives us a nice comparison between the solution $z = c^T x$ of the original problem and the solution $z = b^T y$ of the dual problem.

Weak Duality:

If x and y are feasible solutions to the original and dual problem respectively, then

$$c^T x \leq b^T y$$

Think of this as saying "max $\leq$ min." This says the solution to the dual problem $z = b^T y$ is always bigger than the solution of the original problem $z = c^T x$. So if (referring to the first example), the solution of the dual is $b^T y = 300$, then this tells us that our original z is never more than 300.

Why?

$$c^T x = c \cdot x = x \cdot c = x^T c \leq x^T \left(A^T y\right)$$

(The last step is because $A^T y \geq c$ by the dual problem, and $x \geq 0$)

$$x^T \left(A^T y\right) = \left(x^T A^T\right) y = (Ax)^T y \leq b^T y$$

(The last step is because $Ax \leq b$ by the original problem, and $y \geq 0$)

Putting everything together we then get

$$c^T x \leq b^T y \quad \square$$

Although this is good, it's still problematic! Ideally we want $c^T x = b^T y$, otherwise the dual problem is kind of useless. For example, we want to say that $z = 190$ for the dual problem implies $z = 190$ for the original problem. Luckily there is a stronger version (see below)

That said, we can already get two nice consequences of this:

Corollary:

If the Primal is unbounded ($z = \infty$), then the Dual is infeasible (feasible region is empty)

In particular, if the dual is feasible, then the primal is bounded, which in theory would be a good test for boundedness.

Why? Suppose not. Then by weak duality, we have

$$c^T x \leq b^T y$$

But since $c^T x$ is the solution of a max problem and $b^T y$ is the solution of a min problem, this implies

$$\max c^T x \leq \min b^T y$$

By assumption, $\max c^T x = \infty$, and so $\min b^T y \geq \infty$, which means $b^T y = \infty$ for *all* y in the feasible region of the dual problem, which makes no sense $\Rightarrow\Leftarrow$ $\square$

Co-Corollary:

If the Dual is unbounded, then the Primal is infeasible

Why? If the Dual is unbounded, then by the above, the Dual of the dual is infeasible, and so the Primal is infeasible. $\square$

This is in theory a good test to check if the original problem even has a solution, just check if the dual is unbounded. Notice how the dual here gives us info about the primal.

4. Strong Duality

The good news is that **if** the original problem has a finite solution, then the two solutions coincide:

Strong Duality:

Suppose the primal LP has a finite optimal solution x, then the Dual LP is feasible with a finite optimal solution y, and

$$c^T x = b^T y$$

In particular, an LP has an optimal solution iff its dual has an optimal solution, and the solutions coincide.

So in the context of the first example above, if the original problem is feasible (and bounded) and the solution of the dual problem is $z = 190$, then the solution of the original problem is $z = 190$.

Note: If could happen though that both problems are infeasible!

To see this in action, consider the following:

Example 4:

$$\begin{aligned} \max 3x_1 + 4x_2 \\ \text{subject to } 5x_1 + 6x_2 = 7 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{aligned}$$

(2 variables and 1 constraint becomes 1 variable and 2 constraints)

The dual problem becomes:

$$\begin{aligned} \min 7y_1 \\ \text{subject to } y_1 \text{ unbounded} \\ 5y_1 \geq 3 \\ 6y_1 \geq 4 \end{aligned}$$

But the inequalities give $y_1 \geq \frac{3}{5}$ and $y_1 \geq \frac{4}{6} = \frac{2}{3}$. This implies $y_1 \geq \frac{2}{3}$ and the optimal value here is $y_1 = \frac{2}{3}$, which gives $z = 7y_1 = 7\left(\frac{2}{3}\right) = \frac{14}{3}$.

So *if* the original problem has a finite solution, then it must be $z = \frac{14}{3}$.