LECTURE 9: LP DUALITY (II)

1. Recap

Primal LP Problem:

$$\begin{aligned} &\max c^T x \\ \text{subject to } &Ax \leq b \\ &x \geq 0 \end{aligned}$$

Dual LP Problem:

$$\begin{aligned} &\min b^T y \\ \text{subject to } &A^T y \geq c \\ &y \geq 0 \end{aligned}$$

Weak Duality: "max $\leq$ min"

Strong Duality: "min = max" (if one of them is finite). So if you find that $z = 190$ for the dual, then the answer to the primal is $z = 190$.

2. Possible Scenarios

The following table summarizes all the possibilities that could happen

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Dual \ Primal	Finite Max	Unbounded	Infeasible
Finite Min	✓	X	X
Unbounded	X	X	✓
Infeasible	X	✓	✓

Note: ✓ means "Could happen" whereas X means "Never happens"

The table is read from up to left. For example: "If the Primal has a Finite Max, then it could happen that the Dual has a Finite Min."

Justifications:

- If Primal has a Finite Max, then by Strong Duality, min of Dual = max of Primal (finite by assumption), so the Dual has a Finite Min, so it is **not** unbounded (not infinite) and it's **not** infeasible
- If Primal is unbounded, then max $= \infty$, so by Weak Duality, max $\leq$ min, so min $\geq \infty$ which is impossible. So the Dual is Infeasible, so no Finite Min, and not Unbounded
- If Primal is infeasible but Dual has a finite min, then the Dual of the Dual has a finite max $\Rightarrow$ Primal has a finite max $\Rightarrow$ Primal is feasible $\Rightarrow\Leftarrow$
- There are cases where the Primal is Infeasible but the Dual is Unbounded or Infeasible

Note: This table is symmetric, which follows because the Dual of the Dual is the Primal.

To Summarize: Only one of three things has happen

(1) **Best-Case Scenario:** One problem has a finite answer, then the other one has one too

(2) **Unbounded Scenario:** If one problem is unbounded then the other one is infeasible

(3) **Worst-Case Scenario:** Both problems are infeasible (degenerate case)

3. Application

Example 1:

Suppose you're a farmer who grows x_1 wheat at S_1 dollars/wheat and x_2 barley at S_2 dollars/barley

To grow 1 unit of wheat, need to use 1 unit of land, F_1 units of fertilizer and P_1 units of pesticide, similarly for Barley (with 1, F_2 and P_2)

The total resources available are L land, F fertilizer, and P pesticide.

Decision Variables: x_1 and x_2 (wheat and barley)

Objective Function: $z = S_1x_1 + S_2x_2$

Constraints:

- **Land:** $1x_1 + 1x_2 \leq L$
- **Fertilizer:** $F_1x_1 + F_2x_2 \leq F$

- **Pesticide:** $P_1x_1 + P_2x_2 \leq P$

LP Problem:

$$\begin{aligned} \max z =& S_1x_1 + S_2x_2 \\ \text{Subject to } & x_1 + x_2 \leq L \\ & F_1x_1 + F_2x_2 \leq F \\ & P_1x_1 + P_2x_2 \leq P \\ & x_1, x_2 \geq 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min z =& Ly_1 + Fy_2 + Py_3 \\ \text{Subject to } & y_1 + F_1y_2 + P_1y_3 \geq S_1 \\ & y_1 + F_2y_2 + P_2y_3 \geq S_2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Interpretation of the Dual Problem:

This is really interesting, and explains why the dual is related to the primal.

Suppose you're a planning board that decides on the prices y_L, y_F, y_P of Land, Fertilizer, and Pesticide.

Decision Variables: y_L, y_F, y_P

Objective Function: The board wants to *minimize* the total cost, which is $Ly_L + Fy_F + Py_P$, since we used L land, F fertilizer, and P pesticide.

Constraints: The board sets a *price floor* (minimum price) of S_1 dollars/wheat and S_2 dollars/barley for the farmer

- Per wheat, we need 1 unit of land, F_1 units of fertilizer, and P_1 units of pesticide, so the first price floor becomes

$$y_L + F_1 y_F + P_1 y_P \geq S_1$$

- Similarly for the barley we need $y_L + F_2 y_F + P_2 y_P \geq S_3$

LP Problem:

$$\begin{aligned} \min z = & L y_L + F y_F + P y_P \\ \text{Subject to } & y_L + F_1 y_F + P_1 y_P \geq S_1 \\ & y_L + F_2 y_F + P_2 y_P \geq S_2 \\ & y_L, y_F, y_P \geq 0 \end{aligned}$$

Which is **PRECISELY** the dual problem!!!

So the dual is really looking at the same problem but from a different perspective: While the primal is about a farmer maximizing their profit, the dual is about the planning board minimizing their cost.

Note: Strong Duality kind of says that there is equilibrium, the max of the farmer = min of the planning board, compare this to Demand = Supply in Economics.

4. Complementary Slackness

Recall: We can change a problem $Ax \leq b$ into $Ax = b$ by using slack variables.

Example: If the constraint is $x_1 + 2x_2 \leq 5$, then by defining $s_1 = 5 - x_1 - 2x_2$ we get $x_1 + 2x_2 + s_1 = 5$.

Problem: How do slack variables relate to duality?

Turns out that there is a really suprising relationship!

Primal with Slack:

$$\begin{aligned} \max c^T x \\ \text{subject to } Ax + s = b \\ x, s \geq 0 \end{aligned}$$

Dual with Excess:

$$\begin{aligned} \min b^T y \\ \text{subject to } A^T y - e = c \\ y, e \geq 0 \end{aligned}$$

Here e is called an excess variable. We put $-e$ because we have $A^T y \geq c$, so $e = A^T y - c$, the inequality goes the other way around.

The miracle is that the variable of one problem (x or y) is perpendicular to the slack variable of the other one (e or s)

Complementary Slackness:

$$y \cdot s = 0 \quad \text{and} \quad x \cdot e = 0$$

Application: This allows us to find some values of y if we know what s is, see example below.

Proof: It all boils down to studying the quantity $y^T A x$

On the one hand, since $Ax \leq b$ by the primal problem, we have

$$y^T Ax \leq y^T b$$

On the other hand, since $A^T y \geq c$ by the Dual problem, we get $\left(A^T y\right)^T \geq c^T \Rightarrow y^T A \geq c^T$ and so

$$y^T Ax \geq c^T x$$

$$\text{Hence } c^T x \leq y^T Ax \leq y^T b$$

By **Strong Duality** we have $c^T x = b^T y$ and hence the $\leq$ become $=$. In particular we get $y^T Ax = y^T b$ that is $y^T (b - Ax) = 0$ but since $s = b - Ax$ by definition we get $y^T s = 0$, that is $y \cdot s = 0$

Similar for e if you consider $y^T Ax = c^T x$ and subtract. □

Let's see this in action via an example

Example 2:

$$\begin{aligned} \max \, & 3x_1 + 5x_2 \\ \text{subject to } & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

This is not particularly hard to solve: We get $x_1 = 2$ and $x_2 = 6$ with optimal value $z = 3(2) + 5(6) = 36$

Here the slack variables are

$$\begin{aligned} s_1 &= 4 - x_1 = 4 - 2 = 2 \\ s_2 &= 12 - 2x_2 = 12 - 2(6) = 0 \\ s_3 &= 18 - 3x_1 - 2x_2 = 18 - 3(2) - 2(6) = 0 \end{aligned}$$

So $s = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

Complementary slackness then says that if $y = (y_1, y_2, y_3)$ is the solution to the dual problem, we have

$$s \cdot y = 0 \Rightarrow (2,0,0) \cdot (y_1, y_2, y_3) = 0 \Rightarrow 2y_1 + 0y_2 + 0y_3 = 0 \Rightarrow y_1 = 0$$

So without even *writing* the dual problem, we already know that $y_1 = 0$.

Remark: Notice here that y_1 corresponds to the entry where $s_1 > 0$, and in fact this is always true!

Fact:

$$\begin{aligned} s_i > 0 &\Rightarrow y_i = 0 \\ e_j > 0 &\Rightarrow x_j = 0 \end{aligned}$$

This is what is really meant by complementary slackness! So if $s = (1, 0, 3, 4, 5)$ then we know that $y = (0, \star, 0, 0, 0)$, the zeros to s and y are complementary.

And in fact we can take this further and solve for the dual problem:

Dual Problem:

$$\begin{aligned} &\min 4y_1 + 12y_2 + 18y_3 \\ \text{subject to } & y_1 + 3y_3 \geq 3 \\ & 2y_2 + 2y_3 \geq 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Which now, given $y_1 = 0$ simplifies to

$$\begin{aligned} &\min 12y_2 + 18y_3 \\ \text{subject to } & 3y_3 \geq 3 \\ & 2y_2 + 2y_3 \geq 5 \\ & y_2, y_3 \geq 0 \end{aligned}$$

Which is much simpler, and in fact the minimum values here are $3y_3 = 3 \Rightarrow y_3 = 1$ and $2y_2 = 5 - 2y_3 = 3 \Rightarrow y_2 = \frac{3}{2}$

This gives $y = \left(0, \frac{3}{2}, 1\right)$ and optimal value $z = 4(0) + 12\left(\frac{3}{2}\right) + 18(1) = 18 + 18 = 36$, as expected.

Note: Again, from complementary slackness, we know that since $y_2 > 0$ and $y_3 > 0$, s must be of the form $(\star, 0, 0)$, which true here

Note: We don't even need to calculate e! Since $x = (2, 6)$ with $2 > 0$ and $6 > 0$, this already tells us that $e = (0, 0)$. In fact, notice when we minimized this problem, we set $3y_3 = 3$ and $2y_2 + 2y_3 = 5$, so we made all the constraints tight.