1. (a)

$$\begin{aligned} &\max c^T x \\ \text{subject to } &Ax \leq b \\ &x \geq 0 \end{aligned}$$

where $c = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -3 \\ 1 & 0 & 1 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad b = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$

Note: Another acceptable answer would be

$$c = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -3 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) **Slack Variables:**

$$\begin{aligned} s_1 =& x_1 + 2x_2 - 6 \geq 0 \\ s_2 =& 7 - x_1 - x_2 + x_3 \geq 0 \end{aligned}$$

Standard Form:

$$\begin{aligned} \max\, & x_1 + 3x_2 + 4\left(x_3\right)^+ - 4\left(x_3\right)^- \\ \text{subject to}\, & x_1 + 2x_2 - s_1 = 6 \\ & x_2 + 2\left(x_3\right)^+ - 2\left(x_3\right)^- = 5 \\ & x_1 + x_2 - \left(x_3\right)^+ + \left(x_3\right)^- + s_2 = 7 \\ & x_1, x_2, \left(x_3\right)^-, \left(x_3\right)^+, s_1, s_2 \geq 0 \end{aligned}$$

2. (4 variables with 2 constraints ⇒ 2 variables with 4 constraints)

$$\begin{aligned} \min\ & 3y_1 + 6y_2 \\ \text{subject to}\ & y_1 \geq 2 \\ & 2y_1 + y_2 \geq 4 \\ & 2y_1 + 3y_2 \geq 8 \\ & y_1 + 4y_2 \geq 3 \\ & y_1, y_2 \geq 0 \end{aligned}$$

3.

$$\begin{aligned} \max\ & 2x_1 + 3x_2 \\ \text{subject to } & x_1 + 2x_2 \le 4 \quad \text{①} \\ & 4x_1 + 2x_2 \le 2 \quad \text{②} \\ & x_1 + 3x_2 \le 9 \quad \text{③} \\ & x_1 \ge 0 \quad \text{④} \\ & x_2 \ge 0 \quad \text{⑤} \end{aligned}$$

STEP 1: Starting Vertex

Current Vertex: $(0, 0)$

z-value: $z = 0$

Here ④ and ⑤ are tight.

Since $3 > 2$, we need to increase x_2, so ⑤ is released.

Hitting Times: Set $x_1 = 0$ in each constraint

$$\begin{aligned} &\text{①}\ 0 + 2x_2 = 4 \Rightarrow x_2 = 2 \\ &\text{②}\ 4(0) + 2x_2 = 2 \Rightarrow x_2 = 1 \\ &\text{③}\ 0 + 3x_2 = 9 \Rightarrow x_2 = 3 \end{aligned}$$

Therefore ② is hit first and we hit the vertex $(0, 1)$

STEP 2: Change of Coordinates:

Now ② and ④ are tight

$$\begin{aligned} &\text{④}\ y_1 = x_1 \\ &\text{②}\ 4x_1 + 2x_2 \le 2 \Rightarrow y_2 = 2 - 4x_1 - 2x_2 \end{aligned}$$

$$\begin{cases} y_1 = x_1 \\ y_2 = 2 - 4x_1 - 2x_2 \end{cases}$$

$$x_1 = y_1$$
$$2x_2 = 2 - 4x_1 - y_2 = 2 - 4y_1 - y_2 \Rightarrow x_2 = 1 - 2y_1 - \frac{1}{2}y_2$$

$$\begin{cases} x_1 = y_1 \\ x_2 = 1 - 2y_1 - \frac{1}{2}y_2 \end{cases}$$

Rewrite the objective function in terms of y_1 and y_2

$$z = 2x_1 + 3x_2 = 2y_1 + 3\left(1 - 2y_1 - \frac{1}{2}y_2\right) = 2y_1 + 3 - 6y_1 - \frac{3}{2}y_2 = 3 - 4y_1 - \frac{3}{2}y_2$$

The negative coefficients indicate that we're at a max, and so we stop

STEP 3: Answer:

Optimal Vertex: $(0, 1)$

Optimal $z-$value: $z = 2(0) + 3(1) = 3$

4. **Decision Variables:** x_i which is the weight of coffee sold per day, in pounds/day

   Objective Function: Let I be the set of Golden Blend Coffees

$$z = \sum_{i \in I} (q_i - s_i - r_i t - g_i)\, x_i + \sum_{i \notin I} (q_i - s_i - r_i t)\, x_i$$

   Constraints:

$$\sum_{i=1}^{n} r_i x_i \leq R \text{ Roasting Hours}$$
$$\sum_{i \in I} x_i \leq G \text{ Gold}$$
$$\sum_{i=1}^{n} x_i \geq D \text{ Demand}$$

   LP Problem:

$$\begin{aligned} \max \quad & \sum_{i \in I} (q_i - s_i - r_i t - g_i)\, x_i + \sum_{i \notin I} (q_i - s_i - r_i t)\, x_i \\ \text{subject to} \quad & \sum_{i=1}^{n} r_i x_i \leq R \\ & \sum_{i \in I} x_i \leq G \\ & \sum_{i=1}^{n} x_i \geq D \end{aligned}$$

   Note: Another way of writing the objective function is

$$z = \left(\sum_{i=1}^{n} (q_i - s_i - r_i t)\, x_i \right) - \sum_{i \in I} g_i x_i$$

5. (a) Suppose x and y are in the half space, and $0 \leq \lambda \leq 1$.

   Then $a^T x \leq b$ and $a^T y \leq b$ and so

$$a^T(\lambda x + (1-\lambda)y) = \lambda \underbrace{a^T x}_{\leq b} + (1-\lambda) \underbrace{a^T y}_{\leq b} \leq \lambda b + (1-\lambda)b = b$$

   Hence $a^T(\lambda x + (1-\lambda)y) \leq b$ so $\lambda x + (1-\lambda)y$ is in the half space too.

   (b)

$$f(\lambda x+(1-\lambda)y) = c^T(\lambda x + (1-\lambda y)) = \lambda c^T x+(1-\lambda)c^T y = \lambda f(x)+(1-\lambda)f(y)$$

   In particular, it follows that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

   And so f is convex.