APMA1210 Midterm 1 Practice Exam

Problem 1

Consider the following LP

$$\begin{aligned} \max \quad & z = 3x_1 - 4x_2 + x_3 \\ \text{s.t.} \quad & 2x_1 - x_2 + x_3 \leq 0 \\ & x_1 + 5x_2 - 2x_3 \geq 0 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Put the LP to the standard form.

Solution

The standard form is

$$\begin{aligned} \max \quad & z = 3x_1 - 4x_2 + x_3 \\ \text{s.t.} \quad & 2x_1 - x_2 + x_3 + s_1 = 0 \\ & x_1 + 5x_2 - 2x_3 - s_2 = 0 \\ & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{aligned}$$

Problem 2

The Joy-full Scoop ice cream company has decided to start manufacturing its trademark flavor Cookie Crash close to the stores that sell it, to minimize the cost of shipping for the delicious frozen treat. The company has built m factories, and needs to ship ice cream to n stores. Each store requires d_j containers of Cookie Crash per week, $j = 1, \ldots, n$, and each factory can produce at most p_i containers of Cookie Crash per week, $i = 1, \ldots, m$. It costs c_{ij} dollars to ship 1 container of Cookie Crash from factory i to store j. Construct a linear program that the Joy-full Scoop company could use to minimize its shipping costs while meeting demand and not exceeding production limits.

Solution

We first establish a collection of decision variables x_{ij}, representing the number of containers of Cookie Crash to be shipped from factory i to store j. Then our objective function is $z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$, and we will be working to minimize it. We need constraints for meeting demand and for remaining within production capacity, as well as non-negativity constraints. Be careful of indexing when you construct these! The resulting LP, in summation notation, is as follows:

$$
\begin{aligned}
\text{min} \quad & z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{m} x_{ij} \geq d_j, j = 1, ..., n \\
& \sum_{j=1}^{n} x_{ij} \leq p_i, i = 1, ..., m \\
& x_{ij} \geq 0, i = 1, ..., m, j = 1, ..., n
\end{aligned}
$$

Problem 3

Consider the following LP

$$\begin{aligned} \max \quad & z = 5x_1 + 2x_2 \\ \text{s.t.} \quad & 2x_1 - x_2 \geq 2 \\ & x_1 - 3x_2 \leq 4 \\ & x_1 + x_2 \leq 7 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solve the LP by Simplex method.

Solution

We start from vertex $(1, 0)$. From the objective function, we increase x_1. Let $x_2 = 0$ in the constraints, we have

$$\begin{aligned} 2x_1 - 0 = 2 \Rightarrow x_1 = 1 \quad & \text{(Starting vertex)} \\ x_1 - 3 \times 0 = 4 \Rightarrow x_1 = 4 & \\ x_1 + 0 = 7 \Rightarrow x_1 = 7 & \end{aligned}$$

Then $x_1 - 3x_2 \leq 4$ is hit first. So we introduce the new coordinates

$$y_1 = 4 - x_1 + 3x_2, \quad y_2 = x_2$$

thus

$$x_1 = 4 - y_1 + 3y_2, \quad x_2 = y_2$$

The LP becomes

$$\begin{aligned} \max \quad & z = 20 - 5y_1 + 17y_2 \\ \text{s.t.} \quad & -2y_1 + 5y_2 \geq -6 \\ & -y_1 + 4y_2 \leq 3 \\ & -y_1 + 3y_2 \geq -4 \\ & y_1, y_2 \geq 0 \end{aligned}$$

So we increase y_2. Let $y_1 = 0$ in the constraints, we have

$$\begin{aligned} 2 \times 0 + 5y_2 = -6 \Rightarrow y_2 = -\frac{6}{5} \quad & (\times) \\ 0 + 4y_2 = 3 \Rightarrow y_2 = \frac{3}{4} & \\ 0 + 3y_2 = -4 \Rightarrow y_2 = -\frac{4}{3} \quad & (\times) \end{aligned}$$

Then $-y_1 + 4y_2 \leq 3$ is hit first. Introduce the new coordinates

$$z_1 = y_1, \quad z_2 = y_1 - 4y_2 + 3$$

thus

$$y_1 = \frac{1}{4}(z_1 - z_2 + 3), \quad y_2 = z_2$$

The LP becomes

$$\begin{aligned} \max \quad & z = \frac{131}{4} - \frac{3}{4}z_1 - \frac{17}{4}z_2 \\ \text{s.t.} \quad & \text{Some constraints} \end{aligned}$$

So $z = \frac{131}{4}$ is optimal. The optimal solution is $(\frac{25}{4}, \frac{3}{4})$.

Problem 4

Consider the LP in problem 3, find the dual LP. Change the direction of one of the constraints in problem 3. Will the corresponding dual LP be feasible?

Solution

The original LP is equivalent to

$$\begin{aligned} \max \quad & z = 5x_1 + 2x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 \leq -2 \\ & x_1 - 3x_2 \leq 4 \\ & x_1 + x_2 \leq 7 \\ & -x_1, -x_2 \leq 0 \end{aligned}$$

The corresponding dual LP is

$$\begin{aligned} \min \quad & z = -2y_1 + 4y_2 + 7y_3 \\ \text{s.t.} \quad & -2y_1 + y_2 + y_3 \geq 5 \\ & y_1 - 3y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

If we change the direction of $x_1 + x_2 \leq 7$, then by a theorem in the lecture it follows that the dual is infeasible.