# APMA 1210 Recitation 3

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## 1 Questions

### 1.1 Convexity

In class you were asked to take for granted that the objective function $z = c^T x$ of a linear program will always be convex. Prove this fact, using the definition of convex functions.

### 1.2 Simplex and Graphs

Consider the following linear program:

$$\begin{aligned}
\text{Maximize } & z = x_1 + 2x_2 \\
\text{Subject to } & -x_1 + x_2 \leq 2 \\
& x_1 + x_2 \leq 8 \\
& -x_1 + x_2 \geq -4 \\
& x_1, x_2 \geq 0
\end{aligned}$$

(a) Solve this problem graphically (draw the feasible region and compute the optimal solution).

(b) If you were to solve this problem using the simplex method, you would start at the origin and step from vertex to vertex until the optimal value was found. On your graphical representation, draw the path you think the simplex method would take. Explain your reasoning.

(c) Compute the first step of the simplex method on this linear program. Is it following the path you believed it would take?

## 1.3 Starting vertices

Search for a starting vertex for the following LP, and use your findings to determine whether the LP is bounded:

$$
\begin{aligned}
\text{Maximize } & z = 3x_1 - 5x_2 \\
\text{Subject to } & 2x_1 - x_2 \geq 3 \\
& x_1 + 3x_2 \leq 5 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

# 2 Answers

## 2.1 Convexity

Note that this property is not supposed to depend on how many variables the program has, so you need a proof that works for any number of variables, as well as any coefficient vector $c$. Fortunately, since the objective function $z$ is also linear, this is relatively easy to show. Suppose there are $n \geq 1$ variables, so both the variables and the coefficients are vectors of length $n$. Consider any value of $\lambda \in [0, 1]$, and take any pair of variable vectors $x, y \in \mathbb{R}^n$. Then for the vector $\lambda x + (1 - \lambda)y$, evaluating the objective function at this point will yield $z = c^T(\lambda x + (1 - \lambda)y)$ which gives us the following:

$$\begin{aligned}
c^T(\lambda x + (1-\lambda)y) &= \sum_{i=1}^{n} c_i(\lambda x_i + (1-\lambda)y_i) \\
&= \sum_{i=1}^{n} \lambda c_i x_i + (1-\lambda)c_i y_i \\
&= \lambda \sum_{i=1}^{n} c_i x_i + (1-\lambda)\sum_{i=1}^{n} c_i y_i \\
&= \lambda c^T x + (1-\lambda)c^T y
\end{aligned}$$

Recall that the definition of convex functions requires that $f(\lambda x + (1 - \lambda)y \leq \lambda f(x) + (1 - \lambda)f(y)$. As shown above, the objective function of any linear program will yield equality in this equation, which makes it convex.

## 2.2 Simplex and Graphs

(a) The graphical representation of the feasible region is on the next page. Evaluating the objective function at each of the 5 corners of the feasible region, we find that the optimal value is $z = 13$ at the corner $(3, 5)$.

(b) The arrows on the figure represent the path that the simplex method should take, starting from the origin. Notice that the two possible options starting from the origin yield the same possible increase for $z$ (both vertices correspond to $z = 4$); we choose to go to vertex $(0, 2)$ because it achieves this increase over a shorter edge length. In essence, to make this prediction, we take the value of $z$ at each possible vertex and divide it by the length of the edge connecting it to our starting point. Whichever choice gives the highest ratio of increase in $z$ per unit of distance traveled is the vertex the simplex method will select, as this is the graph principle that corresponds to choosing the vertex with the highest coefficient in the simplex method.

(c) We begin at the origin. Let the first three constraints be numbered (1)-(3) in the order written, and the non-negativity constraints be $x_1 \geq 0$ as (4)

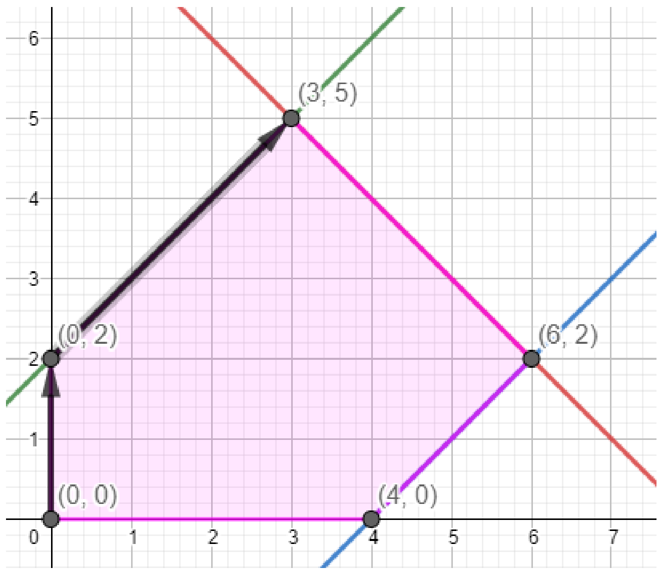


Figure 1: Feasible region and simplex path for problem #2

and $x_2 \geq 0$ as (5)

**Current vertex:** $(x_1, x_2) = (0, 0)$ with tight constraints {(4),(5)}

**Objective value:** $z = 0 + 2(0) = 0$, not optimal; select $x_2$ to increase

**Move:** As $x_2$ increases, constraint (5) releases. Constraint (1) tightens at $x_2 = 2$.

**Coordinates:** Using constraint (1), we find $y_1 = 2 + x_1 - x_2$. Our other tight constraint is (4), yielding $y_2 = x_1$.

**Rewrite LP:** Reorienting our coordinate system, we have $x_1 = y_2$ and $x_2 = 2 - y_1 + y_2$. This yields the following LP:

$$\begin{aligned}
\text{Maximize: } & z = 4 - 2y_1 + 3y_2 \\
\text{Subject to: } & (1) y_1 \geq 0 \\
& (2) - y_1 + 2y_2 \leq 6 \\
& (3) y_1 \leq 6 \\
& (4) y_2 \geq 0 \\
& (5) - y_1 + y_2 \geq -2
\end{aligned}$$

Notice in particular that at the "Move" step, we increased $x_2$ to a value of 2 and did not change the value of $x_1$. This means our first move is to the vertex (0,2) as predicted graphically.

### 2.3 Starting vertices

Following the method outlined in the notes, we define a new LP. Since there are two constraints other than the non-negative constraints, we need two artificial variables $z_1, z_2$. We define the objective function to be $z_1 + z_2$, and we will seek to minimize this function. We create the constraints for this LP by adding $z_1$ and $z_2$ to the left-hand side of each constraint in the original LP, and adding non-negativity constraints for $z_1$ and $z_2$. The resulting LP is as follows:

$$\begin{aligned} \text{Minimize } & z_1 + z_2 \\ \text{Subject to } & 2x_1 - x_2 + z_1 \geq 3 \\ & x_1 + 3x_2 + z_2 \leq 5 \\ & x_1, x_2, z_1, z_2 \geq 0 \end{aligned}$$

For this problem, we can start at the point $x_1 = 0, x_2 = 0, z_1 = 3, z_2 = 5$ which must be a corner of the feasible region. This puts the value of the objective function at 8 and makes the first two constraints tight, as well as the non-negativity constraints for $x_1$ and $x_2$. In the interest of brevity, we will omit the simplex method steps here; you can solve this however you see fit, including using MATLAB. Solving this LP yields an optimal (i.e. minimum) value of 0, which means that the original LP is in fact bounded and therefore has a solution. Moreover, the optimal point is found to be $x_1 = 1.5, x_2 = 0, z_1 = 0, z_2 = 0$ which means that the simplex method can be started at the point $x_1 = 1.5, x_2 = 0$. To do this in practice, you'll want to begin the simplex method by execute a coordinate change to make this starting point look like the "origin" of your coordinate system.