APMA 1210 Recitation 5

October 2022

1 Questions

1.1 Sensitivity Analysis

Consider a generic LP in matrix form:

$$\begin{aligned} \text{Maximize: } & z = c^T x \\ \text{Subject to: } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Suppose that this LP has a unique optimal value. Prove that its shadow prices are exactly the optimal vertex of its dual.

(Hint: consider a vector Δ that makes a *very small* change to the constraints of the primal LP, so that the constraints have the form $Ax \leq b + \Delta$. How does making this change affect the dual?)

1.2 Game Theory

Two companies, Alphabest Inc. and Bookster Ltd., are competing in the market of children's learning-to-read materials. Alphabest has opted to use customer-centric strategies to improve its market share: it gives out discount coupons (A1), offers home delivery services (A2), and includes free gifts with purchases (A3). Bookster prefers standard media advertising: its approaches are targeted internet ads (B1), newspaper inserts (B2), and magazine spreads (B3). The gains matrix is shown below:

	B1	B2	B3
A1	3	-4	2
A2	1	-7	-3
A3	-2	4	7

Write out the LPs for both companies to find their optimal strategies, where Alphabest has the goal of maximizing gains and Bookster has the goal of minimizing losses. Put Alphabest's LP in a form that could be rapidly plugged into MATLAB. Identify the matrices and vectors that show that the problems are dual.

1.3 Intro to Networks

A gas company owns a pipeline network, sections of which are used to pump natural gas from its main field (S) to its distribution center (T). The company has two large pumping stations, A and B, by which the gas can reach the distribution center. At the present time, the company nets 1200 mcf (million cubic feet) of gas per month from its main field and must transport that entire amount to the distribution center. The following are the maximum usage rates and costs associated with each path the gas can take:

	$S \to A$	$S \to B$	$A \to B$	$A \to T$	$B \to A$	$B \to T$
Max use (mcf/month)	500	900	700	400	600	1000
Cost (dollars/mcf)	20	25	10	15	20	40

Draw this network. Include edge directions, label vertices with their supply/demand values, and label all edges with their capacities and costs. Construct the oriented incidence matrix M for this network. Write the LP associated with this problem, specifying what your decision variables mean and what units they have. Put your LP in matrix form and verify that M appears.

2 Solutions

2.1 Sensitivity Analysis

Let's call the optimal value of the original LP z_0. Note that it is the solution both of the original primal LP and its dual, which has the following generic form:

$$\begin{aligned} \text{Minimize: } & z = b^T y \\ \text{Subject to: } & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

Consider some small change to the vector b of constraint values. We express this change as a vector Δ, so that we are now considering the following modified primal LP:

$$\begin{aligned} \text{Maximize: } & z = c^T x \\ \text{Subject to: } & Ax \leq b + \Delta \\ & x \geq 0 \end{aligned}$$

Since z_0 was the unique optimal value of the original LP, this LP will also have a unique optimal value (because we haven't changed the slopes of the objective function or any of the constraints). We denote this value as z_Δ, and observe that it is also the unique optimal value of the modified dual LP:

$$\begin{aligned} \text{Minimize: } & z = (b + \Delta)^T y \\ \text{Subject to: } & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

Note that the constraints of the dual LP have not changed; the only change is in the coefficients of the objective function. Therefore, with a sufficiently small change Δ, the optimal vertex of this modified dual LP will be the same as the optimal vertex of the original dual LP. We designate this vertex as y^*. Then we have $z_0 = b^T y^*$ and $z_\Delta = (b + \Delta)^T y^*$, and in particular we have that the change in the optimal value due to the change Δ to the constraints is $z_\Delta - z_0 = (b + \Delta)^T y^* - b^T y^* = b^T y^* + \Delta^T y^* - b^T y^* = \Delta^T y^*$.

Now we can get specific. Recall that the shadow price of a constraint i is defined as the change to the optimal value caused by increasing b_i by 1. This would correspond to a Δ vector where the i-th entry is 1 and all other entries are 0. For this vector, where we let n be the number of constraints (i.e. the length of b), the change to the optimal value will be $z_\Delta - z_0 = \Delta^T y^* = \sum_{j=1}^{n} \Delta_j y_j^*$, which evaluates as $1 \cdot y_i^* + \sum_{j \neq i} 0 \cdot y_j^* = y_i^*$. So increasing b_i by 1 yields a change of exactly y_i^* in the optimal value, meaning that the i-th entry of the optimal vertex for the dual is the shadow price for the i-th constraint of the primal, and this will hold true for all constraints.

2.2 Game Theory

We will find the LP for Alphabest first. Let Alphabest's strategy be represented by $x = (x_1, x_2, x_3)$ where $x_1 + x_2 + x_3 = 1$ and $x_i \geq 0$. Then the objective for Bookster is to minimize losses, represented by $z = \min\{3x_1 + x_2 - 2x_3, -4x_1 - 7x_2 + 4x_3, 2x_1 - 3x_2 + 7x_3\}$. Note that z can end up being negative here; a negative value for z would indicate that Bookster is experiencing gains and Alphabest is experiencing losses. We can take this minimization function for Bookster and turn it into the constraints for Alphabest's LP, which is the following:

$$\begin{aligned} \text{Maximize: } & z \\ \text{Subject to: } & -3x_1 - x_2 + 2x_3 + z \leq 0 \\ & 4x_1 + 7x_2 - 4x_3 + z \leq 0 \\ & -2x_1 + 3x_2 - 7x_3 + z \leq 0 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

We use the same process to construct the LP for Bookster - determine the objective for Alphabest and turn it into constraints. Let Bookster's strategy be represented by $y = (y_1, y_2, y_3)$ where $y_1 + y_2 + y_3 = 1$ and $y_i \geq 0$. To maximize gains, Alphabest has the objective $z = \max\{3y_1 - 4y_2 + 2y_3, y_1 - 7y_2 - 3y_3, -2y_1 + 4y_2 + 7y_3$. These three expressions become the constraints for Bookster's LP:

$$\begin{aligned} \text{Minimize: } & z \\ \text{Subject to: } & -3y_1 + 4y_2 - 2y_3 + z \geq 0 \\ & -y_1 + 7y_2 + 3y_3 + z \geq 0 \\ & 2y_1 - 4y_2 - 7y_3 + z \geq 0 \\ & y_1 + y_2 + y_3 = 1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

It is easiest to identify the duality from here. In Alphabest's LP, note that we are working with 4 variables, not 3; we consider z to be its own variable, and it will correspond to the fourth row/column of A and the fourth entry for b, c. These are the following:

$$A = \begin{pmatrix} -3 & -1 & 2 & 1 \\ 4 & 7 & -4 & 1 \\ -2 & 3 & -7 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that b and c are the same, so swapping them for the dual problem will yield the same coefficients in the objective function and constants in the constraints; this lines up perfectly with Bookster's equations. Meanwhile, A^T gives the coefficients in the constraints. For the first three constraints, the direction is swapped; for the last one, since it is an equality constraint in the primal problem, the corresponding constraint in the dual must also be an equality constraint. This also aligns with the LP for Bookster, showing that the problems are dual.

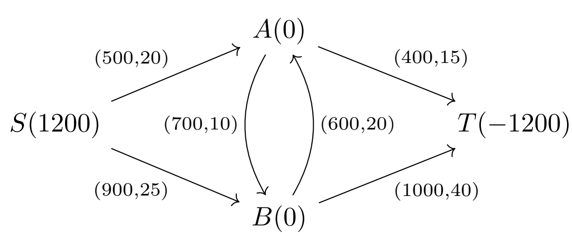
We now move to putting Alphabest's LP in a MATLAB-ready form. This requires us to rewrite the objective function, split the equality constraint into two inequalities, and flip the direction on the non-negativity constraints. Altogether, this results in the following system of equations (written according to the form shown in class):

$$\begin{aligned} \text{Maximize: } & 0x_1 + 0x_2 + 0x_3 - z = 0 \\ \text{Subject to: } & -3x_1 - x_2 + 2x_3 + z \leq 0 \\ & 4x_1 + 7x_2 - 4x_3 + z \leq 0 \\ & -2x_1 + 3x_2 - 7x_3 + z \leq 0 \\ & x_1 + x_2 + x_3 \leq 1 \\ & -x_1 - x_2 - x_3 \leq -1 \\ & -x_1 \leq 0 \\ & -x_2 \leq 0 \\ & -x_3 \leq 0 \end{aligned}$$

The matrices and vectors you would need to use in your MATLAB input are:

$$f = \begin{pmatrix} 0 & 0 & 0 & -1 \end{pmatrix}, A = \begin{pmatrix} -3 & -1 & 2 & 1 \\ 4 & 7 & -4 & 1 \\ -2 & 3 & -7 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2.3 Intro to Networks



The above network includes all labels and edge directions for the problem, including the identification of the field node S, distribution center T, and pumping stations A and B. The oriented incidence matrix for this network is shown below, first as a table so that the rows and columns can be labelled with their corresponding vertices, then as the matrix alone. Make sure to be very precise with your labeling system when constructing these matrices so you know which information goes with which vertices and edges.

	$S \to A$	$S \to B$	$A \to B$	$A \to T$	$B \to A$	$B \to T$
S	1	1	0	0	0	0
A	-1	0	1	1	-1	0
B	0	-1	-1	0	1	1
T	0	0	0	-1	0	-1

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}$$

The decision variables for this problem will be $x_{SA}, x_{SB}, x_{AB}, x_{AT}, x_{BA}, x_{BT}$, where x_{ij} represents the quantity of gas to be sent along the pipeline from node i to node j in mcf/month. Note that we only have decision variables for individual directed edges in the network; if there is not a directed edge from node i to node j, then there is not a decision variable x_{ij} (so for example there is no x_{AS}). Then we can use the network to construct the following LP for minimizing the cost of transporting gas from the field S to the distribution center T:

$$\begin{aligned} \text{Minimize: } & z = 20x_{SA} + 25x_{SB} + 10x_{AB} + 20x_{BA} + 15x_{AT} + 40x_{BT} \\ \text{Subject to: } & x_{SA} + x_{SB} = 1200 \\ & -x_{SA} + x_{AB} - x_{BA} + x_{AT} = 0 \\ & -x_{SB} - x_{AB} + x_{BA} + x_{BT} = 0 \\ & -x_{AT} - x_{BT} = -1200 \\ & x_{SA} \leq 500 \\ & x_{SB} \leq 900 \\ & x_{AB} \leq 700 \\ & x_{BA} \leq 600 \\ & x_{AT} \leq 400 \\ & x_{BT} \leq 1000 \\ & x_{SA}, x_{SB}, x_{AB}, x_{AT}, x_{BA}, x_{BT} \geq 0 \end{aligned}$$

Consider the constraint coefficient matrix A for this LP. Since we have a lot of single-variable constraints, it ends up being very large. To construct it, we need to make sure we use a consistent ordering of the decision variables. Let's use $x_{SA}, x_{SB}, x_{AB}, x_{AT}, x_{BA}, x_{BT}$ like in the objective function. Notice that this matches the edge ordering we created for the oriented incidence matrix. With this ordering, the constraint coefficient matrix appears as follows:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We can see that the first 4 rows of this matrix are precisely the oriented incidence matrix, because each row corresponds to a specific vertex. Our LP lists these in the same order as we used for the oriented incidence matrix, to make this clear, but a reordering of the constraints would not alter the fundamental alignment between the two matrices.