# LECTURE 20: INTEGER PROGRAMMING (II)

**Last time:** Integer Programming (IP), where we assume $x_i$ to be an integer, or even $x_i \in \{0, 1\}$ (binary/decision/yes-no variables)

## 1. "Or" Conditions

While this looks more restrictive at first, here is an example where using IP actually gives us *more* freedom

**Example 1:**

$$\begin{aligned} &\max z \\ \text{subject to } &2x_1 + 3x_2 \geq 4 \\ \underline{\textbf{or}} \ &4x_1 + 5x_2 \geq 7 \\ &x_1, x_2 \geq 0 \end{aligned}$$

In other words (at least) one of the two conditions is satisfied.

The cool thing is you can remove the "or" condition by introducing a new binary variable $y \in \{0, 1\}$

$$\begin{aligned} &\max z \\ \text{subject to } &2x_1 + 3x_2 \geq 4y \\ &4x_1 + 5x_2 \geq 7(1 - y) \\ &x_1, x_2 \geq 0 \\ &y \in \{0, 1\} \end{aligned}$$

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**Why?** Think of $y$ as an off/on switch. If $y = 0$ then $1 - y = 1$ so the second constraint is satisfied, and if $y = 1$ then the first constraint is satisfied. In either case, at least one constraint is satisfied.

Notice here $y + (1 - y) = 1$

We can even generalize this to several constraints.

**Example 2:**

$$\begin{aligned} &\max z \\ \text{subject to } & 2x_1 + 3x_2 \geq 4 \\ & 4x_1 + 5x_2 \geq 7 \\ & 3x_1 + 4x_2 \geq 9 \\ & 8x_1 + 5x_2 \geq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Suppose now you want to say "at least $k = 3$ constraints are satisfied" Then you introduce binary variables $y_1, y_2, y_3, y_4 \in \{0, 1\}$ that sum up to 3 and rewrite your LP as

$$\begin{aligned} &\max z \\ \text{subject to } & 2x_1 + 3x_2 \geq 4y_1 \\ & 4x_1 + 5x_2 \geq 7y_2 \\ & 3x_1 + 4x_2 \geq 9y_3 \\ & 8x_1 + 5x_2 \geq 10y_4 \\ & x_1, x_2 \geq 0 \\ & y_i \in \{0, 1\} \\ & y_1 + y_2 + y_3 + y_4 = 3 \end{aligned}$$

This is because $y_1 + \cdots y_4 = 3$ implies that at exactly three of the $y_i$ are equal to 1, so at least 3 constraints are satisfied.

## 2. Several Values

In similar spirit, what if our variables take on more than 2 values?

**Example 3:**

$$\begin{aligned} &\max z \\ \text{subject to} \quad &\text{Constraints} \\ &x_1 \in \{2, 5, 8\} \\ &x_2 \geq 0 \end{aligned}$$

We can again reduce this to a binary problem by defining new binary variables $y_1, y_2, y_3$ that add up to 1, and add the constraint

$$x_1 = 2y_1 + 5y_2 + 8y_3$$

$$\begin{aligned} &\max z \\ \text{subject to} \quad &\text{constraints} \\ &x_1, x_2 \geq 0 \\ &x_1 = 2y_1 + 5y_2 + 8y_3 \\ &y_i \in \{0, 1\} \\ &y_1 + y_2 + y_3 = 1 \end{aligned}$$

This is because if $y_1 = 0, y_2 = 1, y_3 = 0$ (off/on/off), then $x_1 = 5$, so we indeed get the value 5, and similarly we can get the values 2 and 8 as well.

**Note:** This should be reminiscent of indicator functions in probability and analysis.

## 3. Application: Facility Location

Here's another useful application of Integer Programming

**Example 4:**

You're the mayor of Peyamgeles, and you're trying to figure out where to build hospitals

You can build hospitals at $n = 30$ possible locations and have $m = 1000$ patients (clients) to be serviced.

**Goal:** Figure out where to build the hospitals, and assign each patient to a location, while minimizing a total cost

This can be solved elegantly using binary variables:

### Decision Variables:

First figure out where to build the hospitals. For each location $j$, let

$$y_j = \begin{cases} 1 & \text{if you build a hospital at } j \\ 0 & \text{otherwise} \end{cases}$$

Then figure out if you service patient $i$ at location $j$

$$x_{ij} = \begin{cases} 1 & \text{if patient } i \text{ is serviced at location } j \\ 0 & \text{otherwise} \end{cases}$$

### Objective Function:

The cost of building a hospital at location $j$ is $c_j$

The cost to serve patient $i$ at location $j$ is $d_{ij}$ (think cost of ambulance, or medical bill)

We need to include both costs, so

$$z = \sum_{j=1}^{30} c_j y_j + \sum_{i,j} d_{ij} x_{ij}$$

**Constraints:**

It makes no sense to serve a patient if there is no hospital, so we want to avoid the $1 \leq 0$ scenario, and hence we require

$$x_{ij} \leq y_j \text{ for all } i, j$$

(So if $y_j = 0$ then $x_{ij} = 0$)

Finally, assume each patient gets assigned to exactly one location, so

$$\sum_{j=1}^{30} x_{ij} = 1 \text{ for all } i = 1, \ldots, 1000$$

**IP Problem:**

$$\begin{aligned}
\min z \ &= \sum_{j=1}^{30} c_j y_j + \sum_{i,j} d_{ij} x_{ij} \\
\text{subject to } & x_{ij} \leq y_j \text{ for all } i, j \\
& \sum_{j=1}^{30} x_{ij} = 1 \text{ for all } i = 1, \ldots, 1000 \\
& x_{ij}, y_j \in \{0, 1\}
\end{aligned}$$

**Note:** This has a total of $mn + m = (1000)(30) + 1000 = 31,000$ constraints. We can reduce that number using the following:

**Trick:** If you sum $x_{ij} \leq y_j$ over all $i$, then you get

$$\sum_{i=1}^{1000} x_{ij} \leq \sum_{i=1}^{1000} \underbrace{y_j}_{\text{no } i} = 1000 y_j$$

This is *equivalent* to $x_{ij} \leq y_j$ because in both cases you can show that if $y_j = 0$ then $x_{ij} = 0$ so we're avoiding the $1 \leq 0$ issue.

$$\begin{aligned} \min z \; &= \sum_{j=1}^{30} c_j y_j + \sum_{i,j} d_{ij} x_{ij} \\ \text{subject to } &\sum_{i=1}^{1000} x_{ij} \leq 1000 y_j \text{ for all } j = 1, \cdots, 30 \\ &\sum_{j=1}^{30} x_{ij} = 1 \text{ for all } i = 1, \ldots, 1000 \\ &x_{ij}, y_j \in \{0, 1\} \end{aligned}$$

This new IP uses only $n + m = 30 + 1000 = 1030$ constraints! WOW

## 4. The Geometry of IP Problems

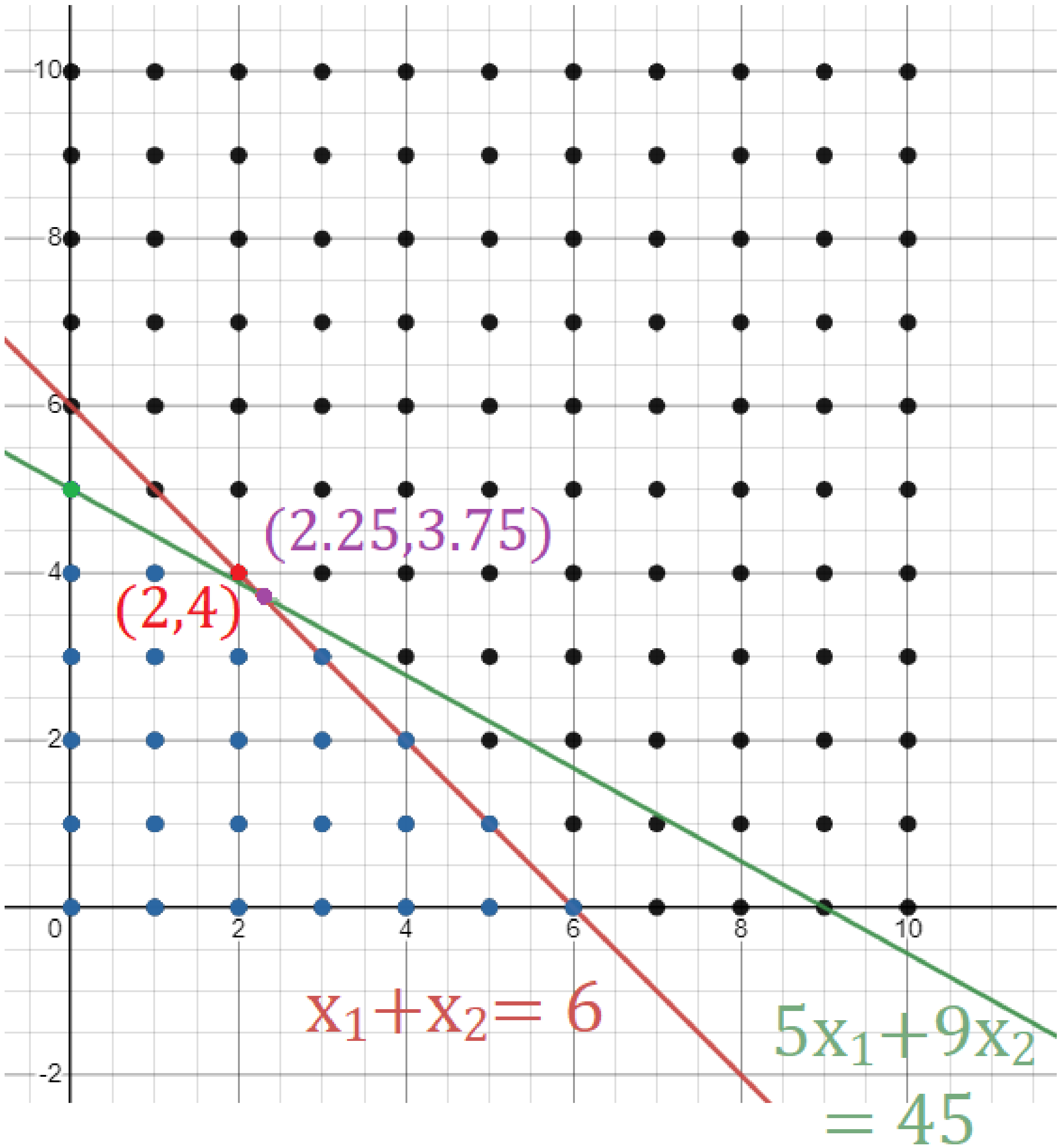
Let's figure out how to solve IP problems, or, rather, how **NOT** to solve IP problems

**Example 5:**

$$\begin{aligned}\max z &= 5x_1 + 8x_2 \\ \text{subject to } & x_1 + x_2 \leq 6 \\ & 5x_1 + 9x_2 \leq 45 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z}\end{aligned}$$

We can draw the region in this case, see picture on the next page.

**Note:** To draw $5x_1+9x_2 = 45$, best to use intercepts. The $x_1$ intercept is $x_1 = 9$ and the $x_2$ intercept is $x_2 = 5$




How would we go about solving this problem?

**Idea:** Hey, let's just ignore the fact that $x_1, x_2 \in \mathbb{Z}$

**Relaxed Problem:** Same as above, but $x_1, x_2 \in \mathbb{R}$.

This now becomes an LP problem.

Solving this, using simplex or by comparing all the vertices, the optimal vertex becomes

$$(x_1, x_2) = \left(\frac{9}{4}, \frac{15}{4}\right) = (2.25, 3.75) \rightsquigarrow z = 5(2.25) + 8(3.75) = 41.25$$

This point is labeled in purple, and is given here by the intersection of the two lines.

**Idea:** Maybe the optimal IP solution is close!

**Round:** $(2.25, 3.75) \approx (2, 4) \rightsquigarrow z = 5(2) + 8(4) = 42$

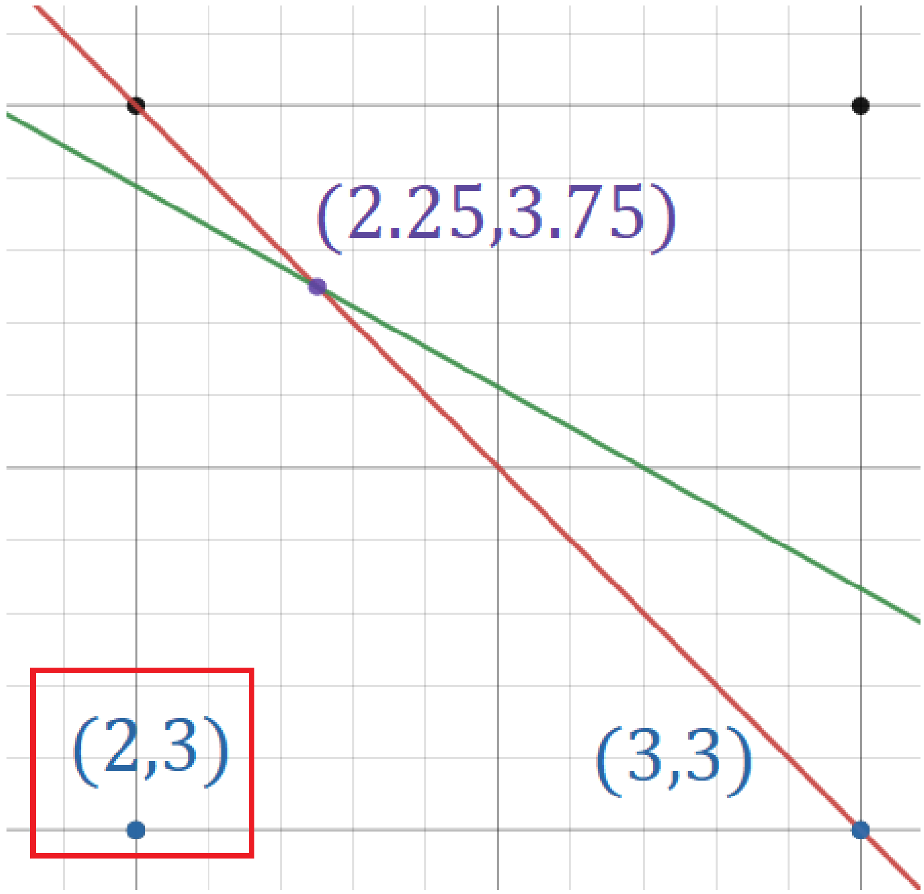
This point is labeled in red.

**Problem:** $(2, 4)$ is **outside** of the feasible region!

In fact, the first constraint is not satisfied because

$$5x_1 + 9x_2 = 5(2) + 9(4) = 46 > 45$$

Ok, so how about we look at the nearest point *in the feasible region*?




There's only two points to consider: $(2,3)$ and $(3,3)$, and the one closest to $(2.25, 3.75)$ in terms of distances is $(2,3)$

$$(2,3) \rightsquigarrow z = 5(2) + 8(3) = 34$$

Is *this* the optimal point? Still no!

In fact, if you consider $(0,5)$, which is feasible, then

$$(0,5) \rightsquigarrow z = 5(0) + 8(5) = 40 \text{ better}$$

This point is labeled in green. In fact, it turns out that $z = 40$ is your optimal value here

**Point:** The optimal IP solution can be far from the relaxed LP sol.

That said, not all is lost!

First of all, notice that the optimal IP value $z = 40$ is $\leq$ the optimal LP value $z = 41.25$, so solving the LP at least gives us an upper bound on our solution.

Moreover, this "zooming-in" feature will lead us to an algorithm called branch-and-bound (next time), a divide-and-conquer algorithm that allows us to solve IP by zooming in and solving mini-LP problems

## 5. Relaxation

To make the preceding discussion more rigorous:

Suppose you want to solve an IP with $x_i \in \mathbb{Z}$

Then the **relaxed** LP is the LP but with $x_i \in \mathbb{R}$

Similarly, if the IP requires $x_i \in \{0, 1\}$, then the **relaxed** LP is the LP with $x_i \in [0, 1]$ or $x_i \in \mathbb{R}$ (depending on the context)

Let $z$ be the optimal IP value and $z^\star$ be the optimal relaxed LP value

(so before we had $z = 40$ and $z^\star = 41.25$)

**Fact: IP $\leq$ LP**

$$z \leq z^{\star}$$

In other words the LP gives us an **upper bound** on our IP solution. In the above problem, without even solving the IP, we know that $z$ is at most 41.25.

**Why?** The IP is more restrictive, so it's harder to get a max that's large. In the relaxed LP, the feasible region is bigger, we have more freedom, so it's easier to get a large max.

And again, those upper bounds will give us a way to solve IP problems, just like the discussions motivating dual problems or min cuts.