APMA1210 Recitation 6

Problem 1

Each of the n teams plays against every other team in a total of k games. Assume that every game ends in a win or a loss (no draws) and let x_i be the number of wins of team i. Let X be the set of all possible outcome vectors $(x_1, x_2, \dots, x_n)$. Given an arbitrary vector $(y_1, y_2, \dots, y_n)$, we would like to determine whether it belongs to X, which is, whether there is a possible tournament outcome vector. Provide a network flow formulation of this problem.

Solution

Take a network with $n + \frac{1}{2}n(n-1)$ nodes, where the first n nodes are the teams and the $\frac{1}{2}n(n-1)$ nodes are the games played between each pair of teams. For each game (i, j), two arcs (ij, i) and (ij, j) are added. Then we get a network.

Now assign each team i with demand y_i and each game (i, j) with source k, we can formulate a network flow problem: let x_{ij} be the flow on the arc (ij, i), we have the flow-balance equations

$$\begin{aligned} &\sum_{j \neq i} x_{ij} = y_i && 1 \leq i \leq n \\ &x_{ij} + x_{ji} = k && 1 \leq i < j \leq n \end{aligned} \tag{1}$$

The capacity constraints:

$$0 \leq x_{ij} \leq k \qquad 1 \leq i \neq j \leq n \tag{2}$$

(Strictly speaking, we also require x_{ij} to be integers.)

$y \in X$ is equivalent to the feasibility of this problem.

Alternative: network with n nodes

We assume each team i has $(n-1)k$ scores at the beginning. If it wins a game with team j, it will gain 1 score from j, say, the score of team i is increased by 1 unit, and the score of team j is deducted by 1 unit. (Equivalently, we are counting the number of wins minus loses.)

In this setting, if a team loses all the games with other $n-1$ teams, its score becomes 0, while if it never loses, it finally gets score $2(n-1)k$.

With this in mind, we construct a network with n nodes representing n teams. Each pair of nodes are connected with an undirected edge.

Given the output $\{y_i\}_{i=1}^n$, each team i finally has score $(n-1)k + y_i - [(n-1)k - y_i] = 2y_i$, so the demand of team i is $2y_i - (n-1)k$. Let x_{ij} be the number wins of team i against team j, then $x_{ij} + x_{ji} = k$. The flow on the arc (i, j) should count the number of scores transported from i to j, which is $f_{ij} = x_{ij} - x_{ji}$ (if $f_{ij} < 0$, this means that the scores are transported from j to i). We only use f_{ij} with $1 \leq i < j \leq n$ as decision variables.

The flow-balance constraints:

$$-\sum_{j<i} f_{ji} + \sum_{j>i} f_{ij} = 2y_i - (n-1)k \qquad 1 \leq i \leq n \tag{3}$$

Capacity constraints:

$$-k \leq f_{ij} \leq k \qquad 1 \leq i < j \leq n \tag{4}$$

$y \in X$ is equivalent to the feasibility of this problem.

Problem 2

Consider the uncapacitated network flow problem shown in Figure 1. The label next to each arc is its cost. Consider the spanning tree indicated by the dashed arcs in the figure and the associated basic solution.

(a) What are the values of the arc flows corresponding to this basic solution? Is this a basic feasible solution?

(b) For this basic solution, find the reduced cost of each arc in the network.

(c) Is this basic solution optimal?

(d) By how much can we increase c_{56} (the cost of arc $(5, 6)$) and still have the same optimal basic feasible solution?

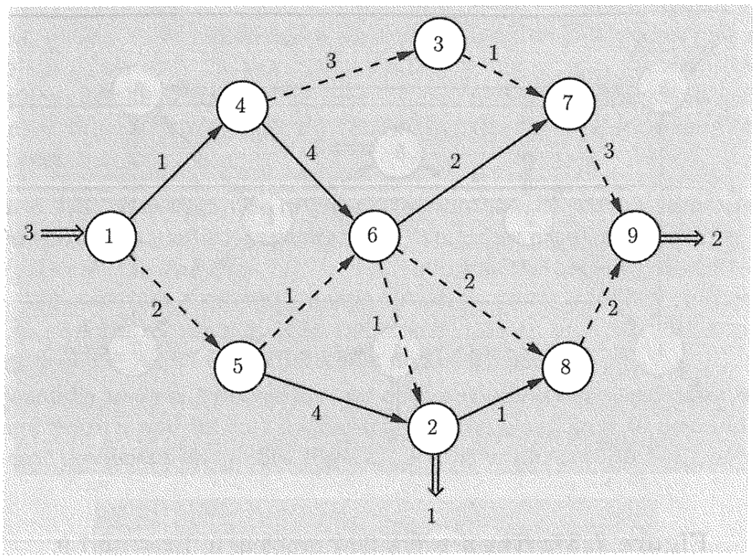

Figure 1: Figure for problem 2

Solution

(a)

See Figure 2.

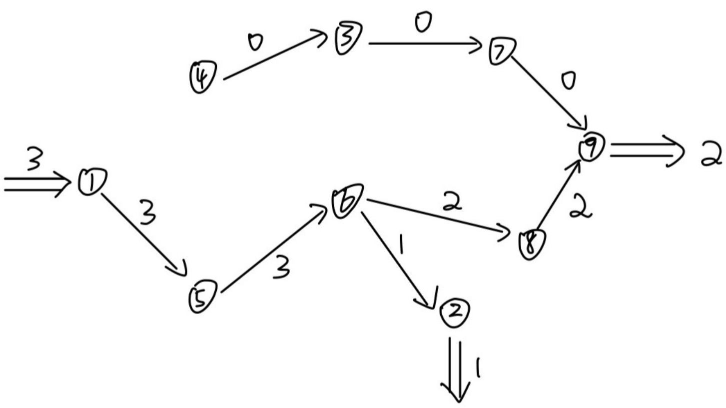

Figure 2: Values of the arc flows.

This is a basic feasible solution as it is a tree solution.

(b)

$$\begin{aligned}
\bar{c}_{14} &= 1+3+1+3-(2+1+2+2) = 2 \\
\bar{c}_{46} &= 4+2+2-(3+1+3) = 1 \\
\bar{c}_{67} &= 2+3-(2+2) = 1 \\
\bar{c}_{52} &= 4-(1+1) = 2 \\
\bar{c}_{28} &= 1+1-2 = 0
\end{aligned}$$

(c)

This is optimal as all the reduced cost is nonnegative.

(d)

$$\begin{aligned}
\bar{c}_{14} &= 1+3+1+3-(2+c_{56}+2+2) = 2-c_{56} \geq 0 \\
\bar{c}_{28} &= 1+c_{56}-2 = c_{56}-1 \geq 0
\end{aligned}$$

Thus $c_{56} \leq 2$. So we can increase c_{56} by 1 at most.