LECTURE 21: BRANCH-AND-BOUND

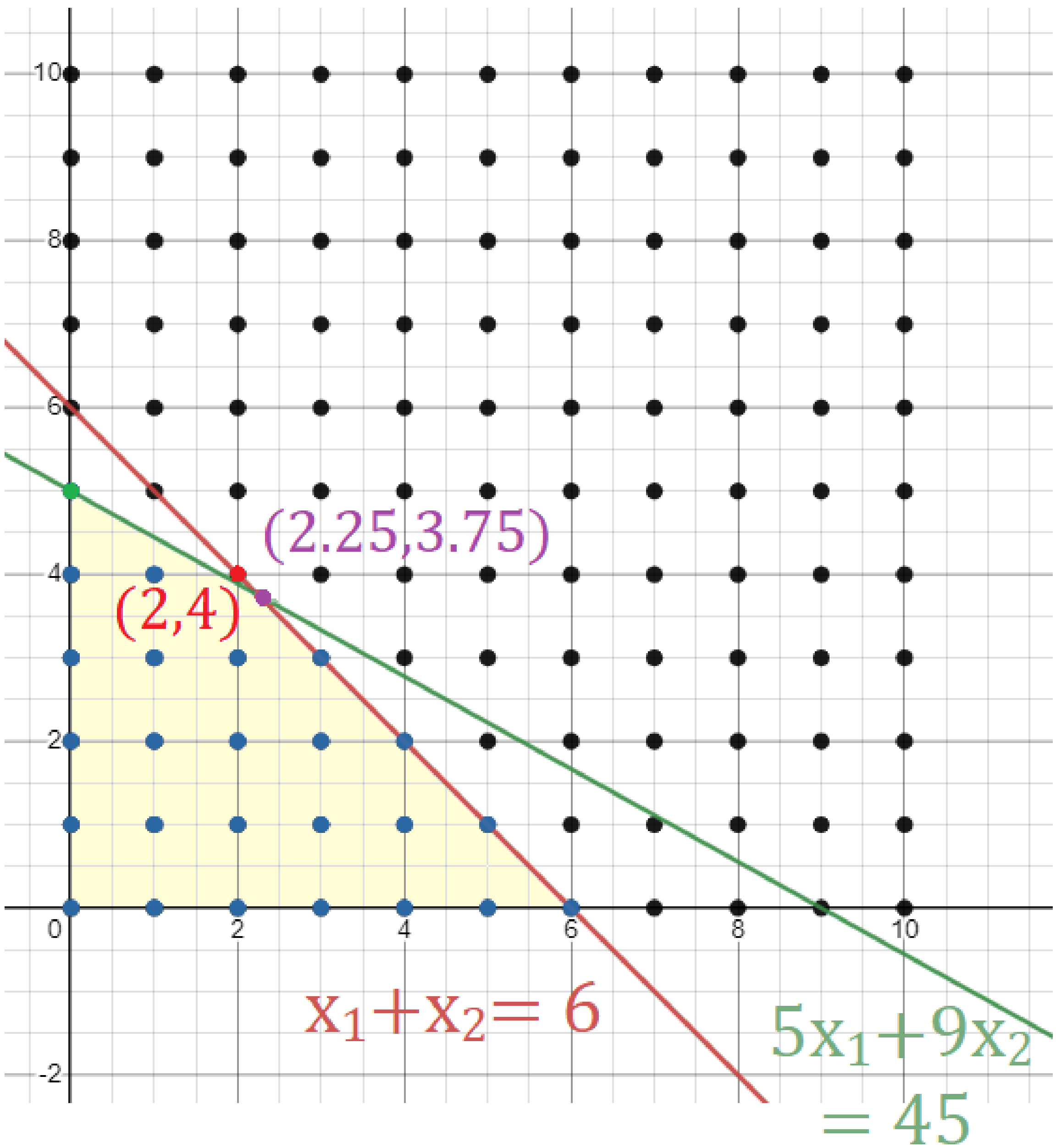
Today: Branch-and-Bound, which is a method to solve IP problems

1. Recap

Example 1:

$$\begin{aligned} \max z &= 5x_1 + 8x_2 \\ \text{subject to } & x_1 + x_2 \leq 6 \\ & 5x_1 + 9x_2 \leq 45 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

Date: Tuesday, November 22, 2022.



Relaxed Problem: Same problem, assume $x_1, x_2 \in \mathbb{R}$

This becomes an LP problem, which you can solve using simplex or by comparing vertices, to get:

$$(x_1, x_2) = \left(\frac{9}{4}, \frac{15}{4}\right) = (2.25, 3.75) \rightsquigarrow z = 41.25$$

(Wrong) Idea: Maybe the optimal IP solution is close!

Round: $(2.25, 3.75) \approx (2, 4) \rightsquigarrow z = 5(2) + 8(4) = 42$

Doesn't work since $(2, 4)$ is **outside** of the feasible region!

Optimal IP Solution: The optimal IP solution turns out to be

$$(0, 5) \rightsquigarrow z = 5(0) + 8(5) = 40$$

Today's Goal: How to find that optimal solution?

Comparison: If $z = 40$ is the optimal IP value and $z^\star = 41.25$ is the optimal (relaxed) LP value, then

Fact: IP $\leq$ LP

$$z \leq z^\star$$

In other words, the LP gives us an **upper bound** on our IP solution. So without even solving the IP, we know that z is at most 41.25.

Remark: Since $z \leq 41.25$ and z is an integer, we actually have $z \leq 41$

2. Branch-and-Bound

Example 2:

Apply Branch-and-Bound to solve the IP above

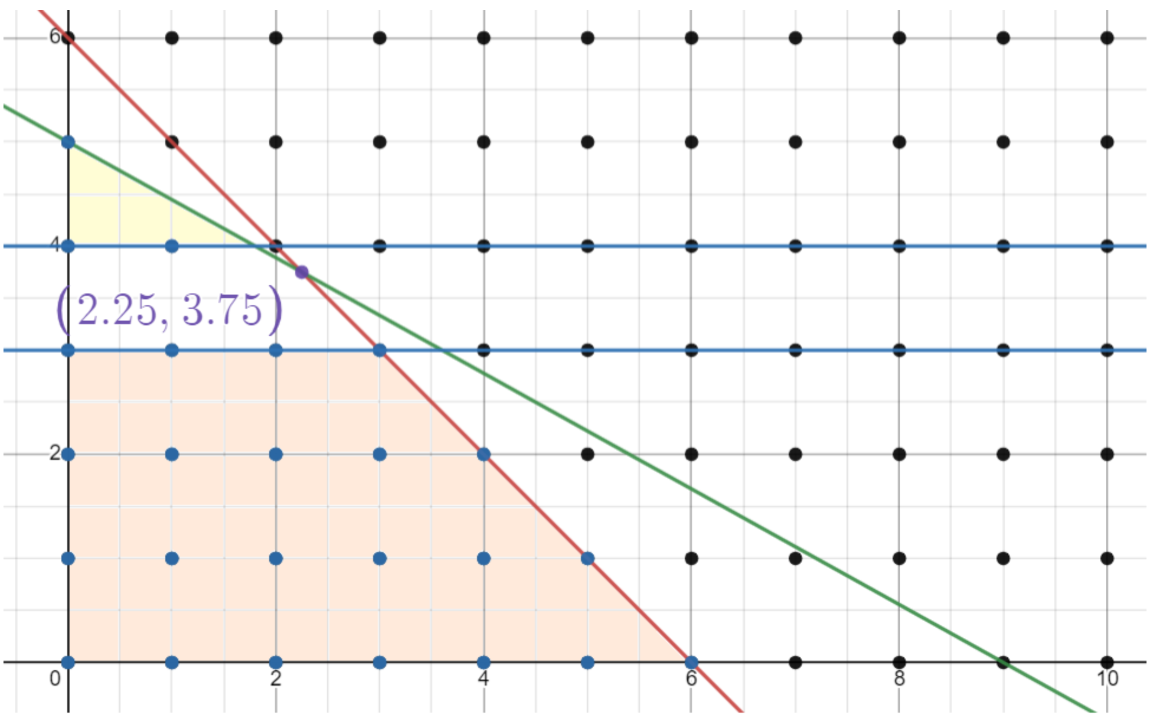
STEP 1: Solve the relaxed LP, where $x_1, x_2 \in \mathbb{R}$

Get optimal vertex $(2.25, 3.75) \Rightarrow x_1 = 2.25$ and $x_2 = 3.75$

Observation: If (x_1, x_2) is the solution of our IP, we need to have either $x_2 \leq 3$ or $x_2 \geq 4$

(The choice of x_2 is arbitrary, perfectly fine to start with x_1)

This splits up our IP problem in two different regions, one where $x_2 \geq 4$ (yellow), and one where $x_2 \leq 3$ (orange) and we just need to solve the problem in each region separately, like a divide-and-conquer algorithm



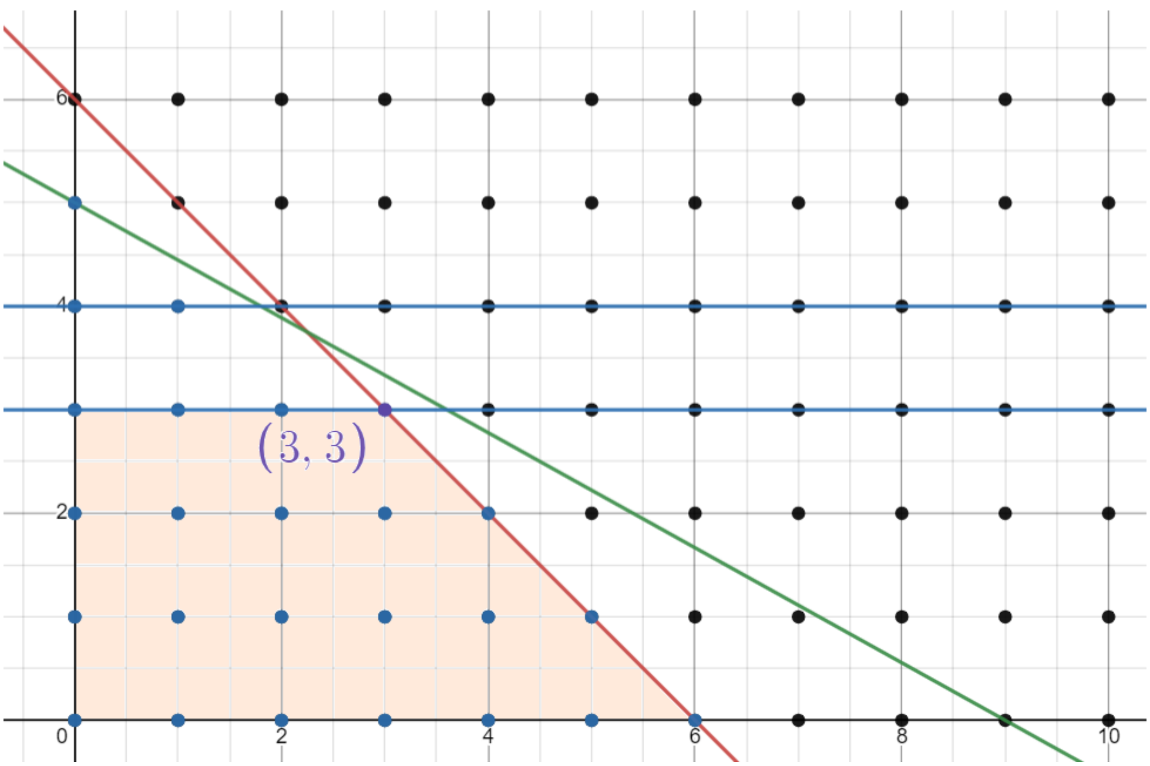
STEP 2:

Case 1: $x_2 \leq 3$ Then solve the LP

$$\begin{aligned} \max z = & 5x_1 + 8x_2 \\ & x_1 + x_2 \leq 6 \\ & 5x_1 + 9x_2 \leq 45 \\ & \color{blue}{x_2 \leq 3} \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{R} \end{aligned}$$

Get optimal vertex $(3, 3)$ and $z = 5(3) + 8(3) = 39$

Here $x_1 = 3$ and $x_2 = 3$, which are both integers



Note: If both x_1 and x_2 are integers, **STOP** do not split up further.

Why? $(3,3)$ is already a solution to the IP problem in that region. So even if you sub-divide the region further, you will either get the same solution, or a solution that's smaller/worse

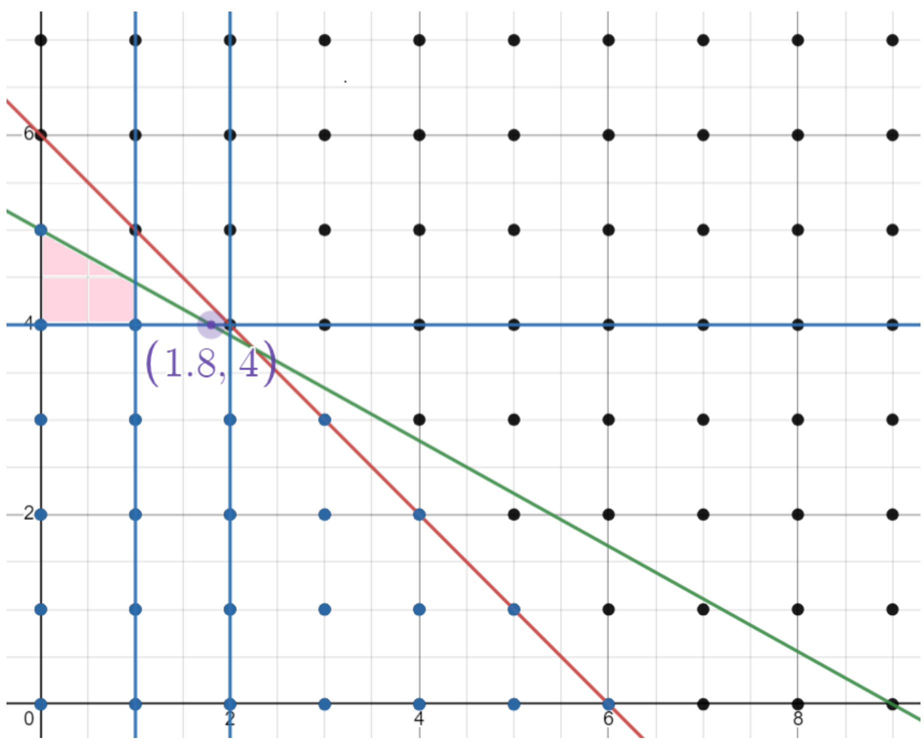
Case 2: $x_2 \geq 4$

In that case, solve the LP with $x_2 \geq 4$ instead of $x_2 \leq 2$

$$\text{Get optimal vertex } (1.8, 4) \text{ and } z = 41$$

Here $x_1 = 1.8$ is not an integer, so we further split up that region into $x_1 \leq 1$ and $x_1 \geq 2$

STEP 3:



Case 2a: $x_1 \geq 2$

In that case the LP is unfeasible, because if $x_1 \geq 2$ and $x_2 \geq 4$, then

$$5x_1 + 9x_2 \geq 5(2) + 9(4) = 10 + 36 = 46 \not\leq 45$$

In that case **STOP**, do not sub-divide further

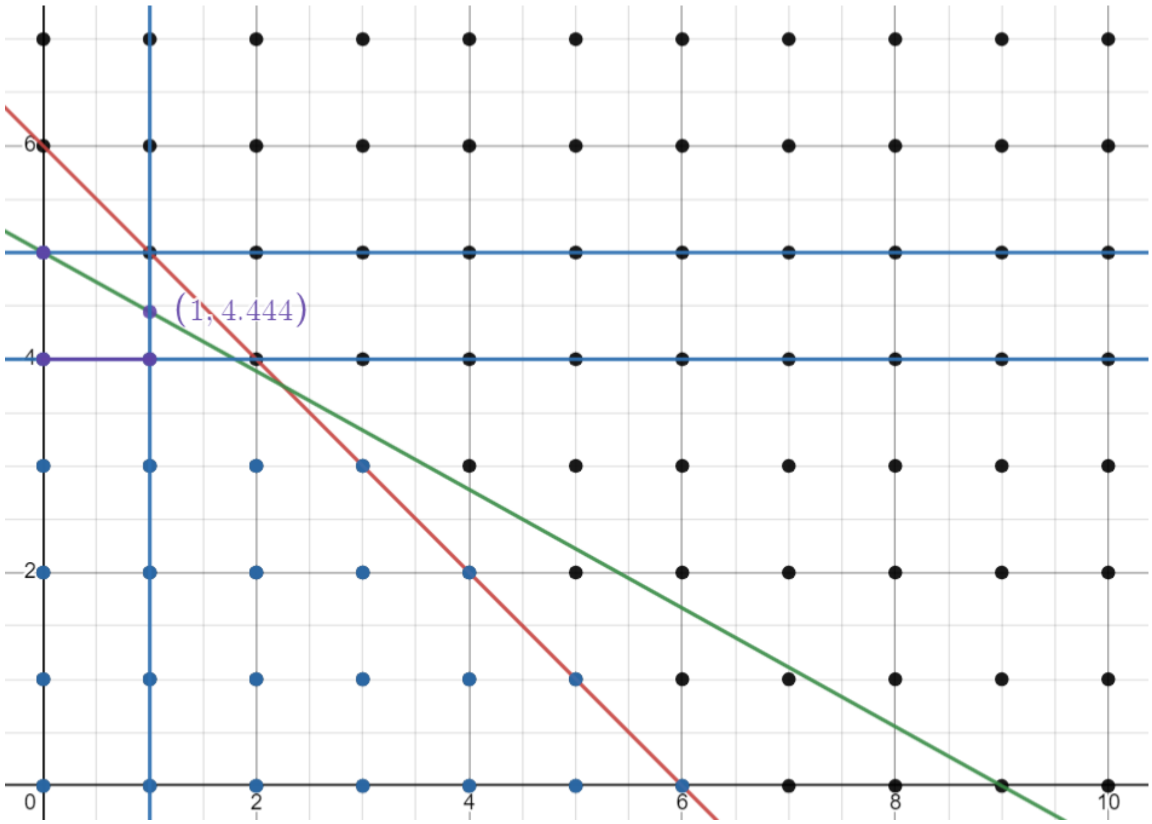
Case 2b: $x_1 \leq 1$ then solve the LP

$$\begin{aligned} \max z = & 5x_1 + 8x_2 \\ & x_1 + x_2 \leq 6 \\ & 5x_1 + 9x_2 \leq 45 \\ & x_2 \geq 4 \\ & x_1 \leq 1 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{R} \end{aligned}$$

$$\text{Get optimal vertex } \left(1, \frac{40}{9}\right) \approx (1, 4.44) \Rightarrow z \approx 40.55$$

x_2 is not an integer, so sub-divide further into $x_2 \leq 4$ and $x_2 \geq 5$

STEP 3:



Case 2b(i): $x_2 \leq 4$. Then we need to solve the LP

$$\begin{aligned}\max z = & 5x_1 + 8x_2 \\ & x_1 + x_2 \leq 6 \\ & 5x_1 + 9x_2 \leq 45 \\ & x_2 \geq 4 \\ & x_1 \leq 2 \\ & x_2 \leq 4 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{R}\end{aligned}$$

This is much easier because $x_2 \geq 4$ and $x_2 \leq 4$ becomes $x_2 = 4$, so we get a 1 variable LP problem, which gives us

$$\text{Optimal Vertex } (1,4) \text{ and } z = 37$$

Since both components are integers, we **STOP**

Case 2b(ii): $x_2 \geq 5$. Then we need to solve the LP

$$\begin{aligned}
\max z = & 5x_1 + 8x_2 \\
& x_1 + x_2 \leq 6 \\
& 5x_1 + 9x_2 \leq 45 \\
& x_2 \geq 4 \\
& x_1 \leq 2 \\
& x_2 \geq 5 \\
& x_1, x_2 \geq 0 \\
& x_1, x_2 \in \mathbb{R}
\end{aligned}$$

The $x_2 \geq 4$ constraint is redundant here, so solving that LP gives

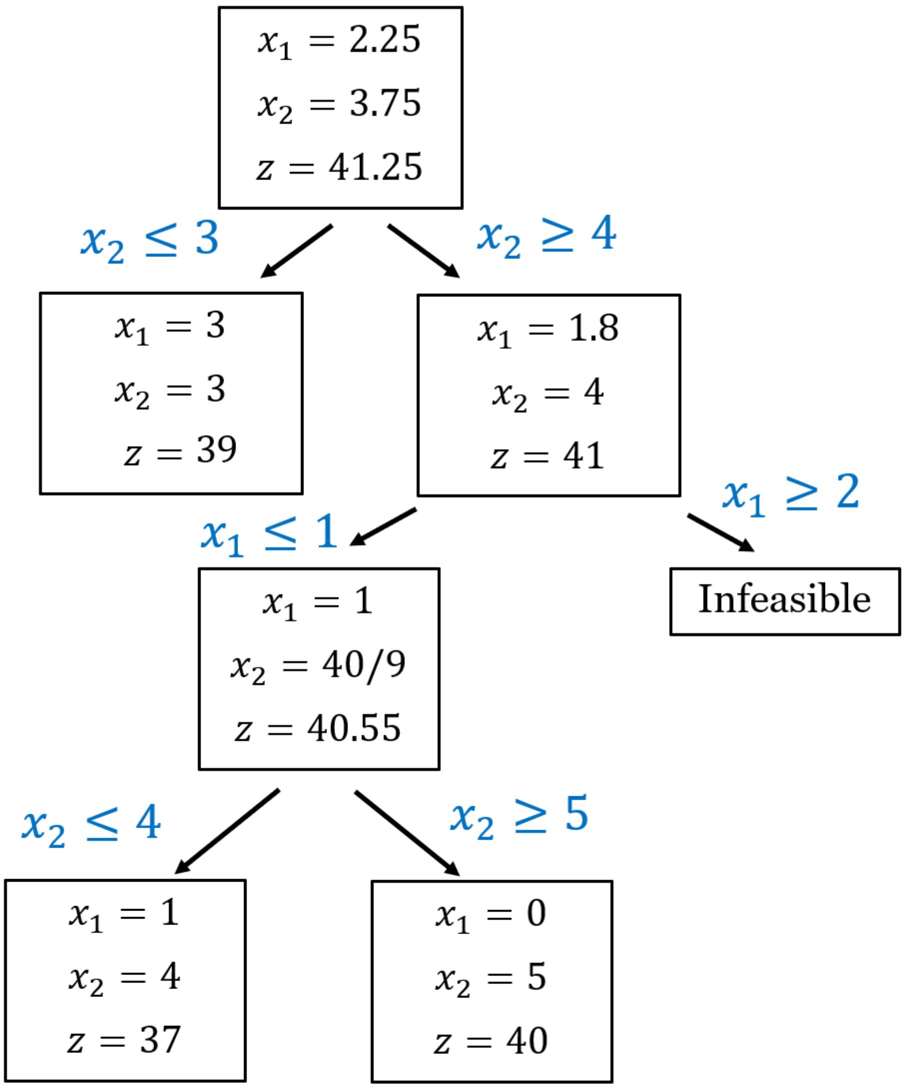
$$\text{Optimal Vertex } (0,5) \text{ and } z = 40$$

Since we have integer components, we **STOP** this case

We ran out of regions altogether, so we **STOP** the algorithm

STEP 4: Compare

It's helpful to draw a flowchart with all our findings:



In the end, we get the following candidates:

$$\begin{array}{ll} (3,3) & z = 39 \\ (1,4) & z = 37 \\ (0,5) & z = 40 \end{array}$$

Now just pick the vertex that gives the biggest value:

Answer: $(0, 5)$ with $z = 40$

3. Remarks

Two remarks are in order:

Efficiency: At first glance, it *looks* like we need to solve a lot of LP problems, but in practice it's not that bad. If you do this by hand by comparing vertices, you'll notice that there are many of them that you'll already have checked. This is because in each LP problem, we're only adding/removing one constraint. This is similar in spirit to when we did shadow prices, where we changed one constraint at a time.

Partial Solutions: What if you're *really* pressed on time and don't solve all the LP problems? Imagine for instance your boss telling you to give an answer ASAP. Then you can still get partial information about this.

Example: Suppose you *only* solve the original LP, and the one that gives you $(1, 4)$. From the original LP and the fact that IP $\leq$ LP, we know that $z \leq 41$. But now, since $(1, 4)$ is *an* integer solution, we also know that $z \geq 37$, by definition of max z.

Hence $37 \leq z \leq 41$, this tells us our optimal z is either $37, 38, 39, 40, 41$, which gives us a better range of our possible z values.

In that case, you would tell your boss that $(1, 4)$ is a solution, and $z = 37$ is a value that is within 4 of our optimal z value. So even if we don't know what our optimal value is, we know we're at most 4 away from it. Alternatively, you can say that we're within $\frac{41-37}{37} = 11\%$ of our optimal $z-$value.

4. "Strong" Formulation

Recall: Our hospital problem, where we decided to build hospitals y_j and assign patients x_{ij} to hospitals:

Example 3: (Hospital IP Problem)

$$\begin{aligned} &\min z \\ \text{subject to } & x_{ij} \leq y_j \\ & \sum_{j=1}^{30} x_{ij} = 1 \text{ for all } i = 1, \ldots, 1000 \\ & x_{ij}, y_j \in \{0, 1\} \end{aligned}$$

Relax this problem by requiring $x_{ij}, y_j \in [0, 1]$

Let P be the feasible region of that relaxed problem.

However, we saw that this problem used $31,000$ constraints, so to reduce this number to 1030, we replaced $x_{ij} \leq y_j$ by

$$\sum_{i=1}^{1000} x_{ij} \leq 1000 y_j$$

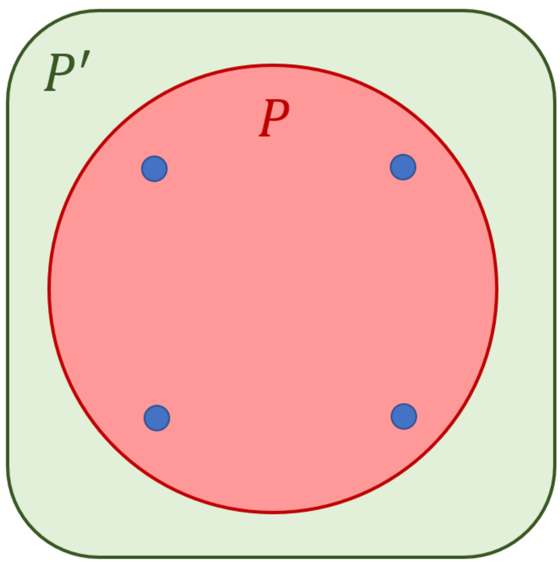
Consider the IP with that replaced constraint, and relax once again by requiring $x_{ij}, y_j \in [0, 1]$

Let P' be the feasible region of this new relaxed problem.

Carefully note: While both problems are the same in terms of integer programming (they have the same integer solutions), in terms of relaxed LP problems, P' is more general than P. In fact, since P' uses fewer constraints (= more freedom) we have

$$P \subseteq P'$$

Which is problematic because P' might give us points that are *far* from the optimal solution!



Surprisingly, less is not more: While using fewer constraints speeds up your simplex algorithm, it does *not* always make your problems better.

In this sense, we say that the $x_{ij} \leq y_j$ formulation is **stronger** than the $\sum x_{ij} \leq 1000 y_j$ one: Even though they represent the same IP problem, the former has a smaller LP feasible region.

Note: Ideally, we want P to be as small as possible, so the ideal scenario would be if $P = \text{conv}(S)$ where $\text{conv}(S)$ the convex hull of all the integer points S in the IP problem. In that case we say that the formulation is **ideal**