LECTURE 25: REVIEW

1. Simplex Method

Example 1:

$$\begin{aligned} \max z = & 2x_1 + 3x_2 \\ \text{subject to } & -x_1 + x_2 \leq 3 \quad ① \\ & x_1 - 2x_2 \leq 2 \quad ② \\ & 3x_1 + 4x_2 \leq 26 \quad ③ \\ & x_1 \geq 0 \quad ④ \\ & x_2 \geq 0 \quad ⑤ \end{aligned}$$

Picture: (optional, see next page)

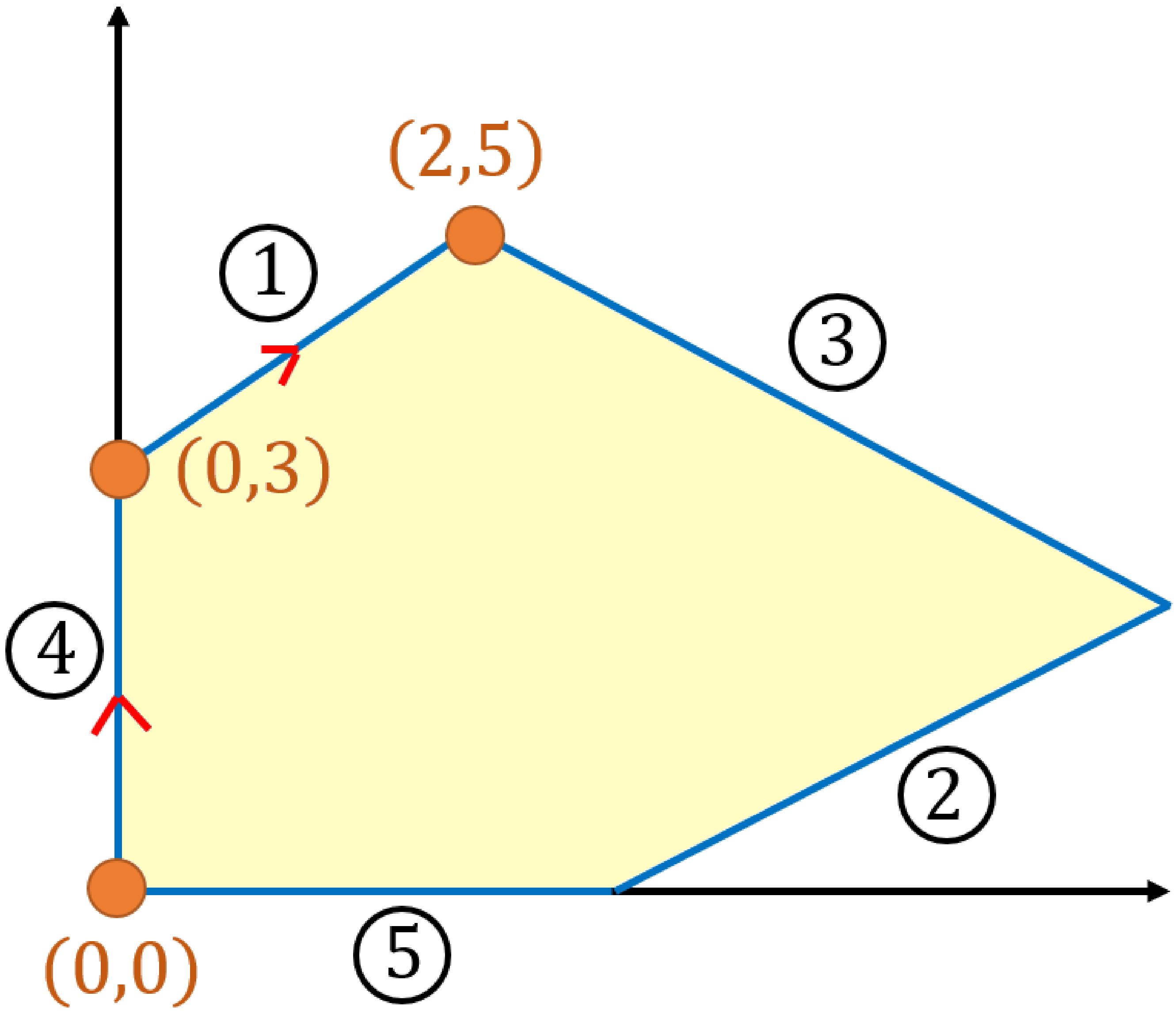
STEP 1: Start at $(0,0)$

Current Vertex: $\{$④, ⑤$\}$

Objective Value: $z = 0$

Because $3 > 2$, increase x_2, so hold ④ and release ⑤

Date: Thursday, December 8, 2022.



Hitting times: Here $x_1 = 0$

$$\begin{aligned}
&① \ -0 + x_2 = 3 \Rightarrow x_2 = 3\\
&② \ 0 - 2x_2 = 2 \Rightarrow x_2 = -1 \times\\
&③ \ 0 + 4x_2 = 26 \Rightarrow x_2 = 6.5
\end{aligned}$$

The smallest hitting time is $x_2 = 3$, so ① is hit first

$$\textbf{New Vertex: } \{④, ①\} = (0, 3)$$

Coordinates:

$$\begin{aligned}
&④ \ y_1 = x_1\\
&① \ -x_1 + x_2 \leq 3 \Rightarrow 3 + x_1 - x_2 \geq 0 \Rightarrow y_2 = 3 + x_1 - x_2
\end{aligned}$$

$$\begin{cases} y_1 = x_1 \\ y_2 = 3 + x_1 - x_2 \end{cases}$$

Change coordinates:

$$\begin{aligned}
x_1 &= y_1\\
x_2 &= 3 + x_1 - y_2 = 3 + y_1 - y_2
\end{aligned}$$

Rewrite problem:

$$\begin{aligned}
\max\ z =& 2x_1 + 3x_2 = 2y_1 + 3(3 + y_1 - y_2) = 9 + 5y_1 - 3y_2\\
&① \ y_2 \geq 0\\
&② \ y_1 - 2(3 + y_1 - y_2) \leq 2 \Rightarrow -y_1 + 2y_2 - 6 \leq 2 \Rightarrow -y_1 + 2y_2 \leq 8\\
&③ \ 3y_1 + 4(3 + y_1 - y_2) \leq 26 \Rightarrow 7y_1 - 4y_2 + 12 \leq 26 \Rightarrow 7y_1 - 4y_2 \leq 14\\
&④ \ y_1 \geq 0\\
&⑤ \ 3 + y_1 - y_2 \geq 0 \Rightarrow -y_1 + y_2 \leq 3
\end{aligned}$$

$$\begin{aligned} \max\ z = & 9 + 5y_1 - 3y_2 & \\ \text{subject to } & y_2 \geq 0 & \textcircled{1} \\ & -y_1 + 2y_2 \leq 8 & \textcircled{2} \\ & 7y_1 - 4y_2 \leq 14 & \textcircled{3} \\ & y_1 \geq 0 & \textcircled{4} \\ & -y_1 + y_2 \leq 3 & \textcircled{5} \end{aligned}$$

STEP 2: $(0, 3)$

Current Vertex: $\{\textcircled{1}, \textcircled{4}\}$

Objective Value: $z = 9$

Because of 5, we increase y_1, so hold $\textcircled{1}$ and release $\textcircled{4}$

Hitting times: Here $y_2 = 0$

$$\begin{aligned} &\textcircled{2}\ -y_1 + 2(0) = 8 \Rightarrow y_1 = -8 \times \\ &\textcircled{3}\ 7y_1 - 0 = 14 \Rightarrow y_1 = 2 \\ &\textcircled{5}\ -y_1 + 0 = 3 \Rightarrow y_1 = -3 \times \end{aligned}$$

The smallest hitting time is $y_1 = 2$, so $\textcircled{3}$ is hit first

New Vertex: $\{\textcircled{1}, \textcircled{3}\} = (2, 0)$ in $y-$coordinates

Coordinates:

$$\begin{cases} \textcircled{3}\ z_1 = 14 - 7y_1 + 4y_2 \\ \textcircled{1}\ z_2 = y_2 \end{cases}$$

Change coordinates:

$$
\begin{aligned}
y_2 =& z_2 \\
y_1 =& -\frac{1}{7}z_1 + \frac{4}{7}y_2 + 2 = -\frac{1}{7}z_1 + \frac{4}{7}z_2 + 2
\end{aligned}
$$

Rewrite problem:

$$
z = 9 + 5y_1 - 3y_2 = 9 + 5\left(-\frac{1}{7}z_1 + \frac{4}{7}z_2 + 2\right) - 3z_2 = 19 - \frac{5}{7}z_1 - \frac{1}{7}z_2
$$

Since both coefficients are negative, we **STOP**

STEP 3: Answer

Optimal z−value: $z = 19$

Optimal Vertex:

In y−coordinates the vertex is $(2, 0)$ and so $y_1 = 2$ and $y_2 = 0$ and so in x-coordinates this becomes

$$
\begin{aligned}
x_1 =& y_1 = 2 \\
x_2 =& 3 + y_1 - y_2 = 3 + 2 - 0 = 5
\end{aligned}
$$

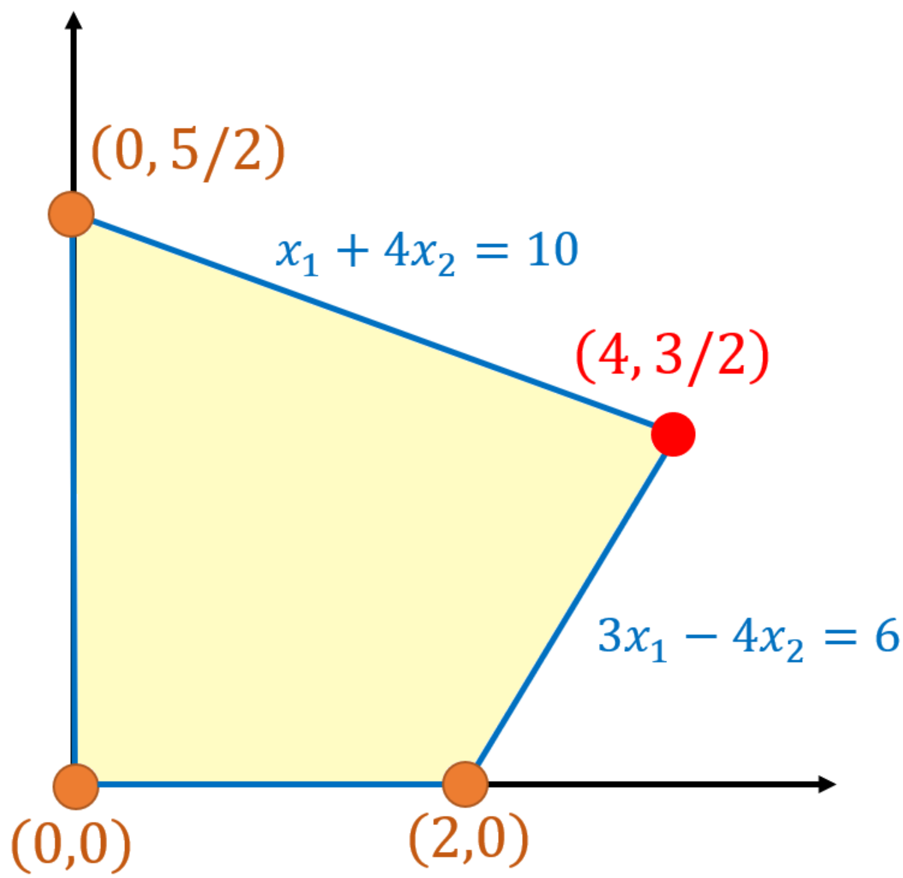
And so the optimal vertex is $(2, 5)$

2. Branch-and-Bound

Example 2:

$$\begin{aligned} \max z = & 4x_1 + 5x_2 \\ \text{subject to } & x_1 + 4x_2 \leq 10 \\ & 3x_1 - 4x_2 \leq 6 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

Picture: Optional, but really useful



STEP 1: Solve the (relaxed) LP problem

Vertices:

$(0,0)$

x_1 intercept of $3x_1 - 4x_2 = 6$ which is $3x_1 - 0 = 6$ so $x_1 = 2 \Rightarrow (2,0)$

x_2 intercept of $x_1 + 4x_2 = 10$ which is $0 + 4x_2 = 10$ so $x_2 = \frac{5}{2} \Rightarrow \left(0, \frac{5}{2}\right)$

Intersection of the two lines

$$\begin{cases} x_1 + 4x_2 = 10 \\ 3x_1 - 4x_2 = 6 \end{cases}$$

Adding the two equations we get $4x_1 = 16$ so $x_1 = 4$ and so $4x_2 = 10 - x_1 = 10 - 4 = 6$ and so $x_2 = \frac{3}{2} \Rightarrow \left(4, \frac{3}{2}\right)$

Compare:

$(0,0)$ with $z = 0$

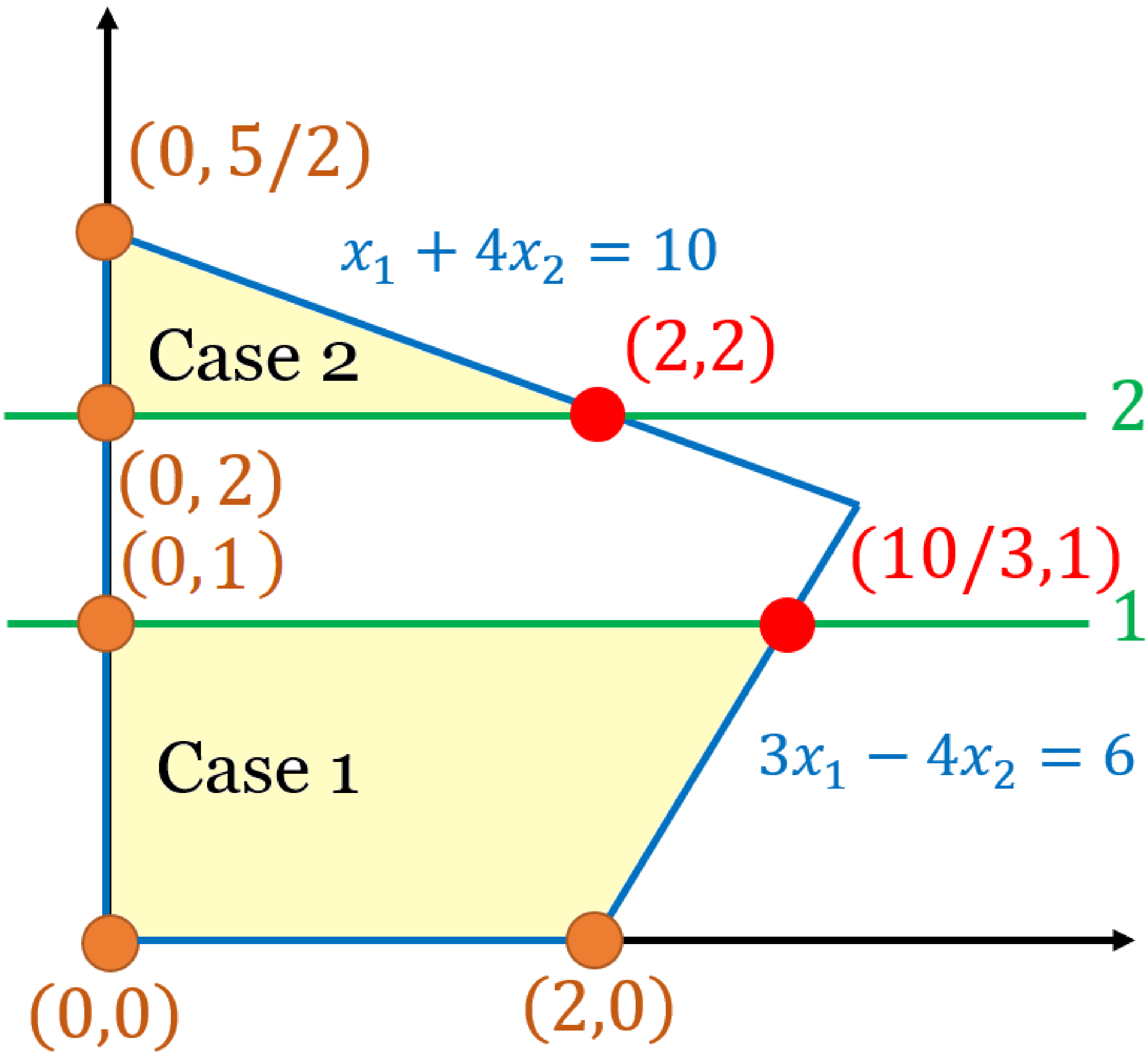
$(2,0)$ with $z = 4(2) + 5(0) = 8$

$\left(0, \frac{5}{2}\right)$ with $z = 4(0) + 5\left(\frac{5}{2}\right) = \frac{25}{2} = 12.5$

$\left(4, \frac{3}{2}\right)$ with $z = 4(4) + 5\left(\frac{3}{2}\right) = 16 + \frac{15}{2} = 16 + 7.5 = 23.5$

The optimal relaxed LP vertex is $\left(4, \frac{3}{2}\right)$ with $z = 23.5$

STEP 2: Since $x_2 = \frac{3}{2}$, we need to split into two cases and solve the relaxed LP in each case



Case 1: $x_2 \leq 1$

Vertices:

$(0,0)$ with $z=0$

$(2,0)$ with $z=8$

$(0,1)$ with $z=4(0)+5(1)=5$

Intersection of $3x_1-4x_2=6$ with $x_2=1$ which gives $3x_1-4(1)=6$ so $3x_1=10$ so $x_1=\frac{10}{3}\Rightarrow\left(\frac{10}{3},1\right)$ with $z=4\left(\frac{10}{3}\right)+5(1)=\frac{65}{3}\approx 18.33$

This last one gives us the biggest z value, therefore

The optimal relaxed LP vertex is $\left(\frac{10}{3},1\right)$ with $z=\frac{65}{3}$

Since $x_1=\frac{10}{3}$, need to split into two sub-cases, see picture below.

Sub-Case 1(a): $x_1\leq 3$

Vertices

$(0,0)$ with $z=0$

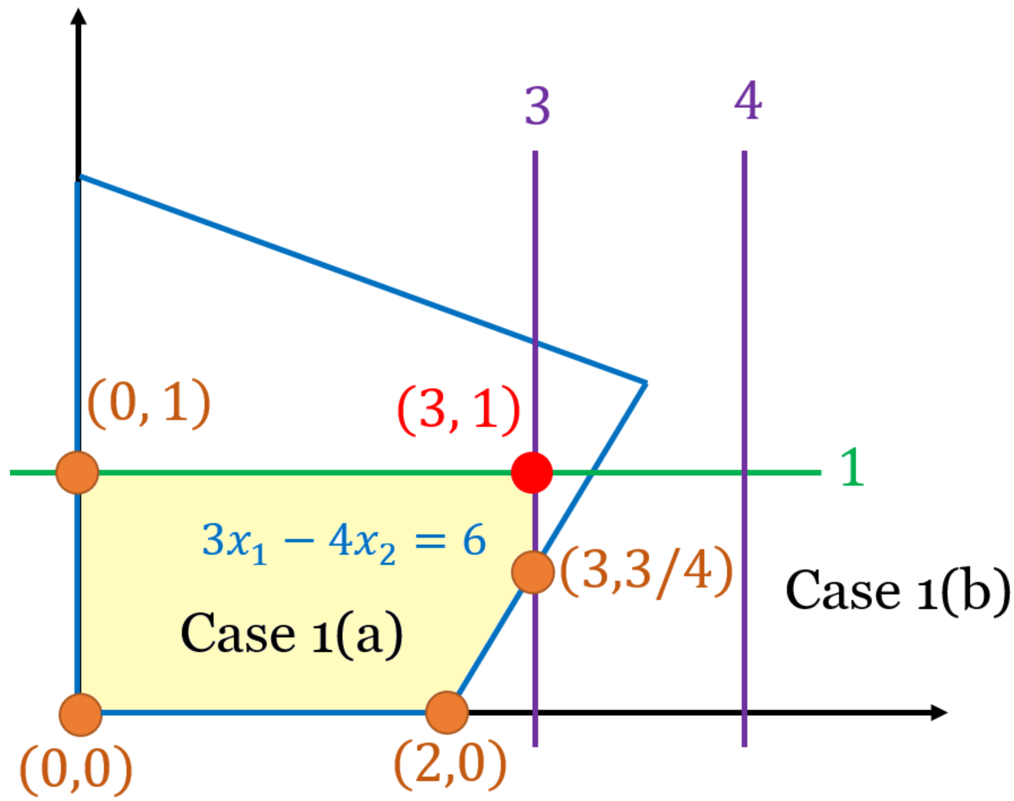
$(2,0)$ with $z=8$

$(0,1)$ with $z=5$

$(3,1)$ with $z=4(3)+5(1)=17$

Intersection of $3x_1-4x_2=6$ with $x_1=3$ which is $3(3)-4x_2=6$ so $-4x_2=-3$ so $x_2=\frac{3}{4}\Rightarrow\left(3,\frac{3}{4}\right)$ with $z=4(3)+5\left(\frac{3}{4}\right)=\frac{63}{4}=15.75$

The optimal relaxed LP vertex is $(3,1)$ with $z=17$



Sub-Case 1(b): $x_1 \geq 4$

This is outside of our feasible region, because if $x_1 \geq 4$ and $x_2 \leq 1$

$$3x_1 - 4x_2 \geq 3(4) - 4(1) = 8 > 6$$

So the second constraint isn't satisfied.

Case 2: $x_2 \geq 2$

Vertices

$\left(0, \frac{5}{2}\right)$ with $z = \frac{25}{2}$

$(0,2)$ with $z = 4(0) + 5(2) = 10$

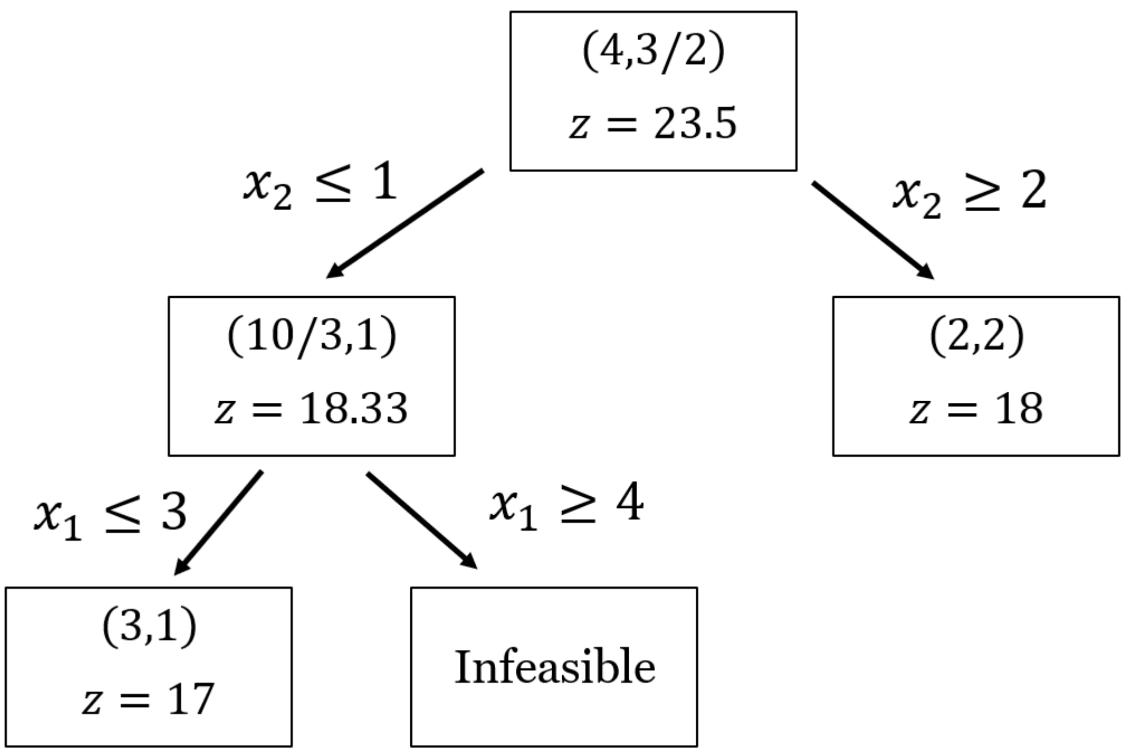
Intersection of $x_2 = 2$ and $x_1 + 4x_2 = 10$ which is $x_1 + 4(2) = 10$ so $x_1 = 2 \Rightarrow (2,2)$ with $z = 4(2) + 5(2) = 18$

The optimal relaxed vertex is $(2,2)$ with $z = 18$

STEP 3: We therefore get 2 candidates:

$(3,1)$ with $z = 17$ and $(2,2)$ with $z = 18$, therefore:

Answer: $(2,2)$ with $z = 18$



3. Concavity

Example 3:

For which values of c is $f(x_1, x_2)$ convex? concave?

$$f(x_1, x_2) = e^{3(x_1)+c(x_2)^2}$$

STEP 1: Calculate $D^2 f$

$$\begin{aligned} f_{x_1} =& 3e^{3(x_1)+c(x_2)^2} = 3f \\ f_{x_2} =& (2cx_2)\, e^{3(x_1)+c(x_2)^2} = (2cx_2)\, f \end{aligned}$$

$$\begin{aligned} f_{x_1x_1} =& 3\,(3f) = 9f \\ f_{x_1x_2} =& (3f)_{x_2} = 3\,(2cx_2)\, f = (6cx_2)\, f \\ f_{x_2x_1} =& (2cx_2 f)_{x_1} = (2cx_2)\,(3f) = (6cx_2)\, f \checkmark \\ f_{x_2x_2} =& (2cx_2 f)_{x_2} = 2cf + (2cx_2)\, f_{x_2} = 2cf + (2cx_2)\, 2cx_2 f \end{aligned}$$

$$D^2 f(x_1, x_2) = \begin{bmatrix} 9f & (6cx_2)\, f \\ (6cx_2)\, f & \left(2c + 4c^2\,(x_2)^2\right) f \end{bmatrix}$$

STEP 2: Convexity: Want $D_2 > 0$ and $D_1 > 0$

$$\begin{aligned} D_2 =& \begin{vmatrix} 9f & (6cx_2)\, f \\ (6cx_2)\, f & \left(2c + 4c^2\,(x_2)^2\right) f \end{vmatrix} \\ =& 9f \left(2c + 4c^2\,(x_2)^2\right) f - (6cx_2)\, f\,(6cx_2)\, f \\ =& \left[18c + \cancel{36c^2\,(x_2)^2} - \cancel{36c^2\,(x_2)^2}\right] f^2 \\ =& 18cf^2 \end{aligned}$$

Since $f > 0$ we have $D_2 > 0$ if and only if $c > 0$

$$D_1 = \det[9f] = 9f$$

Which is always positive

Conclusion: f is convex whenever $c > 0$

STEP 3: Concavity

Either re-do the above but with $-f$ instead of f

Or just check that $(-1)^2 D_2 > 0$ and $(-1) D_1 > 0$, that is $D_2 > 0$ and $D_1 < 0$.

$D_2 = 18cf^2 > 0$ if and only if $c > 0$ but $D_1 = 9f$ which is never negative!

Conclusion: f is never concave

Example 4:

Show that if f is convex, then

$$f\left(\frac{x+y}{2}\right) \leq \left(\frac{1}{2}\right) f(x) + \left(\frac{1}{2}\right) f(y)$$

So for convex f we have f of the midpoint (average) of x and y is less than or equal to the midpoint (average) of $f(x)$ and $f(y)$

$$f\left(\frac{x+y}{2}\right) = f\left(\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)y\right) \leq \left(\frac{1}{2}\right) f(x) + \left(\frac{1}{2}\right) f(y)$$

In the middle step we used the definition of convexity with $\lambda = \frac{1}{2}$, which implies $1 - \lambda = 1 - \frac{1}{2} = \frac{1}{2}$