APMA1210 Recitation 9

Problem 1

Consider the integer programming problem:

$$\begin{aligned} \max z &= x_1 + 2x_2 \\ \text{s.t.} \quad & -3x_1 + 4x_2 \leq 4 \\ & 3x_1 + 2x_2 \leq 11 \\ & 2x_1 - x_2 \leq 5 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

(a) What is the optimal cost of the linear programming relaxation? What is the optimal cost of the integer programming problem?

(b) What is the convex hull of all solutions to the integer programming problem?

(c) Solve the problem by branch and bound. Solve the linear programming relaxations graphically.

Solution

(a)

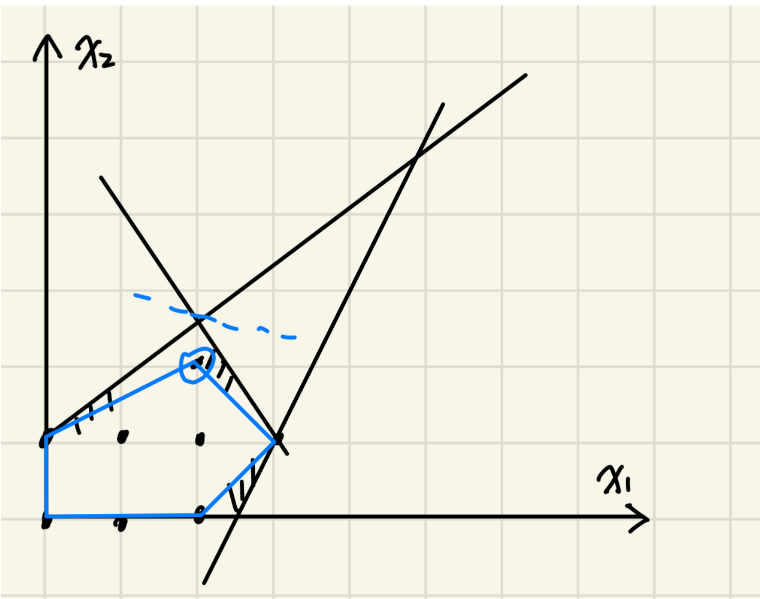

Figure 1: LP & IP

In Figure 1, the feasible region of LP is the shadowed area, and the feasible region of IP is the black dots. The optimal solution of the LP relaxation is $(2, \frac{5}{2})$, and the value is 7. By enumeration, the optimal solution of IP is $(2, 2)$, and the value is 6.

(b)

The convex hull of the IP is shown in Figure 1 with the blue polygon.

(c)

The optimal solution of the LP relaxation is $(2, \frac{5}{2}$, so we start from x_2.
1. Either $x_2 \leq 2$ or $x_2 \geq 3$. But $x_2 \geq 3$ is out of the feasible region, so we only need to consider $x_2 \leq 2$. Adding this to LP relaxation, we have the new optimal solution of LP is $\frac{7}{3}, 2$.
2. For x_1, either $x_1 \leq 2$ or $x_1 \geq 3$. For $x_1 \geq 3$, we have only one feasible solution for LP, that is $(3, 1)$. The value is 5. For $x_1 \leq 2$, the LP relaxation has the optimal solution $(2, 2)$ with value 6, which is greater than 5. So the optimal solution to IP is $(2, 2)$ with value 6.

Problem 2

Compute the longest increasing subsequence (LIS) of the sequence $s = 1, 5, 3, 9, 6, 4, 7, 8$.

Solution

We solve the problem by using dynamic programming. Key observation: the LIS of the subsequence ending with s_x is determined by all LIS ending before s_x.
First, we assume the length of the LIS ending with s_x is $f(x)$. Then the length of the LIS of the entire sequence is $\max_x f(x)$.
For $p < x$, if $s_p < s_x$, then $f(x) \geq f(p) + 1$, thus $f(x) = \max_{p<x,\ s_p<s_x} f(p) + 1$ (state transition). So we can start from $f(1)$.

$$
\begin{aligned}
&f(1) = 1 \\
&f(2) = f(1) + 1 = 2 \\
&f(3) = \max\{f(1)\} + 1 = 2 \\
&f(4) = \max\{f(1), f(3)\} + 1 = 3 \\
&f(5) = \max\{f(1), f(2), f(3)\} + 1 = 3 \\
&f(6) = \max\{f(1), f(3)\} + 1 = 3 \\
&f(7) = \max\{f(1), f(2), f(3), f(5), f(6)\} + 1 = 4 \\
&f(8) = \max\{f(1), f(2), f(3), f(5), f(6), f(7)\} + 1 = 5
\end{aligned}
$$

We can trace the LIS backward, that is

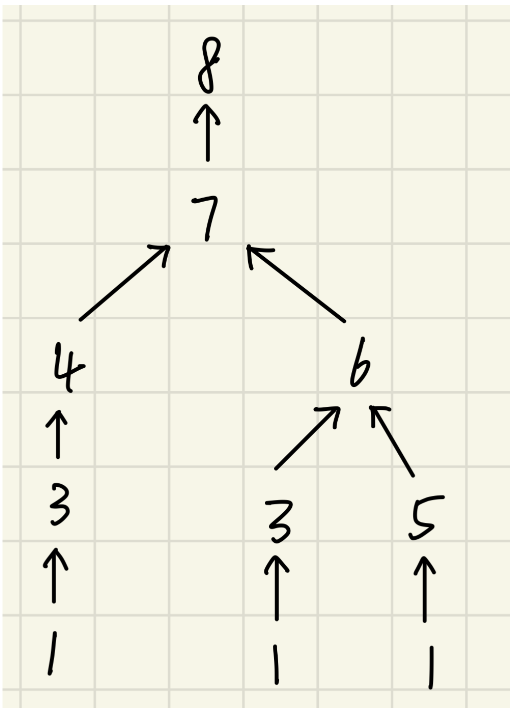

Figure 2: The LIS of sequence s. There are more than one LIS.

Problem 3

Find a path from the upper left corner of the matrix to the lower right corner, s.t. the summation of the values on the path (cost of the path) is minimized. Only one step right/down is allowed each time. Here is the matrix

$$A = \begin{pmatrix} 3 & 1 & 8 & 9 \\ 6 & 8 & 4 & 7 \\ 8 & 3 & 3 & 6 \\ 7 & 1 & 7 & 6 \end{pmatrix}$$

Solution

Since the cost of the path ending at (m, n) is determined by the cost of the path ending at $(m-1, n)$ and $(m, n-1)$, we can apply dynamic programming in a recursive way.
Assume the cost of the path ending at (m, n) is $f(m, n)$, then we have the state transition

$$f(m, n) = \min\{f(m-1, n), f(m, n-1)\} + a_{mn}$$

We can assume that f takes value $+\infty$ on the left boundary and the upper boundary, so we obtain the cost matrix $f(A)$ as

$$f(A) = \begin{pmatrix} 3 & 4 & 12 & 21 \\ 9 & 12 & 16 & 23 \\ 17 & 15 & 18 & 24 \\ 24 & 16 & 23 & 29 \end{pmatrix}$$

Then we can get the optimal path

$$(4, 4) \leftarrow (4, 3) \leftarrow (4, 2) \leftarrow (3, 2) \leftarrow (2, 2) \leftarrow (1, 2) \leftarrow (1, 1)$$