SOLUTIONS

1. *(10 points)* Solve the following equation and write your answer in explicit form

$$\begin{cases} y' + \cos(t)y = te^{-\sin(t)} \\ y(0) = 4 \end{cases}$$

STEP 1 INTEGRATING FACTOR : $e^{\int \cos(t)dt} = e^{\sin(t)}$

STEP 2 $e^{\sin(t)}\left(y' + \cos(t)y\right) = \cancel{e^{\sin(t)}}\, t\, \cancel{e^{-\sin(t)}}$

$\left(e^{\sin(t)} y\right)' = t \Rightarrow e^{\sin(t)} y = \frac{t^2}{2} + C$

$y = e^{-\sin(t)}\left(\frac{t^2}{2} + C\right)$

STEP 3 INITIAL CONDITION : $y(0) = 4$

$\underbrace{e^{-\sin(0)}}_{1}\left(\frac{0^2}{2} + C\right) = 4 \Rightarrow C = 4$

STEP 4 $y = e^{-\sin(t)}\left(\frac{t^2}{2} + 4\right)$

$\boxed{y = e^{-\sin(t)}\left(\frac{t^2}{2} + 4\right)}$

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2. *(10 points)* Use separation of variables to solve the following. Write your answer in explicit form.

$$\begin{cases} y' = \dfrac{y^2+1}{x^2+1} \\ y(0) = 1 \end{cases}$$

STEP 1

$$\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$$

$$\int \frac{dy}{y^2+1} = \int \frac{dx}{x^2+1}$$

$$\tan^{-1}(y) = \tan^{-1}(x) + C$$

$$y = \tan\left(\tan^{-1}(x) + C\right)$$

STEP 2

INITIAL CONDITION

$$y(0) = 1 \implies \tan\left(\tan^{-1}(0) + C\right) = 1$$

$$\implies \tan(C) = 1$$

$$\implies C = \tan^{-1}(1) = \frac{\pi}{4}$$

STEP 3

$$y = \tan\left(\tan^{-1}(x) + \frac{\pi}{4}\right)$$

$$\boxed{y = \tan\left(\tan^{-1}(x) + \frac{\pi}{4}\right)}$$

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3. *(10 points)* Use undetermined coefficients to solve

$$\begin{cases} y'' + 9y = \overset{5}{\text{[redacted]}}\cos(2t) \;\text{[redacted]}\overset{-10}{}\sin(2t) \\ y(0) = 3 \\ y'(0) = \text{[redacted]}\ 5 \end{cases}$$

STEP 1 HOMOGENEOUS SOLUTION

AUX $r^2 + 9 = 0 \implies r = \pm 3i$

$y_0(t) = A\cos(3t) + B\sin(3t)$

STEP 2 UNDETERMINED COEFFICIENTS

$5\cos(2t) - 10\sin(2t) \rightsquigarrow r = \pm 2i$; NO RESONANCE

$y_p(t) = A\cos(2t) + B\sin(2t)$

$y_p'(t) = -2A\sin(2t) + 2B\cos(2t)$

$y_p''(t) = -4A\cos(2t) - 4B\sin(2t)$

$y_p'' + 9y_p = 5\cos(2t) - 10\sin(2t)$

$-4A\cos(2t) - 4B\sin(2t) + 9A\cos(2t) + 9B\sin(2t) = 5\cos(2t) - 10\sin(2t)$

$5A\cos(2t) + 5B\sin(2t) = 5\cos(2t) - 10\sin(2t)$

$$\begin{cases} 5A = 5 \\ 5B = -10 \end{cases} \implies \begin{cases} A = 1 \\ B = -2 \end{cases}$$

$$\boxed{y = 2\cos(3t) + 3\sin(3t) + \cos(2t) - 2\sin(2t)}$$

$y_p(t) = 1\cos(2t) - 2\sin(2t)$

$y_p(t) = \cos(2t) - 2\sin(2t)$

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STEP 3

$$y = y_0 + y_p$$

$$y = A\cos(3t) + B\sin(3t) + \cos(2t) - 2\sin(2t)$$

STEP 4

INITIAL CONDITIONS

$$y(0) = 3$$

$$A\cos(0) + B\sin(0) + \cos(0) - 2\sin(0) = 3$$

$$A + 1 = 3$$

$$A = 2$$

$$y'(t) = -3A\sin(3t) + 3B\cos(3t) - 2\sin(2t) - 4\cos(2t)$$

$$y'(0) = -3A\sin(0) + 3B\cos(0) - 2\sin(0) - 4\cos(0)$$

$$y''(0) = 3B - 4 = 5$$

$$3B = 9$$

$$B = 3$$

$$y = 2\cos(3t) + 3\sin(3t) + \cos(2t) - 2\sin(2t)$$

4. *(10 points)* Use Laplace transforms to solve

$$\begin{cases} y'' - 7y' + 12y = \delta(t-3) \star \delta(t-5) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

STEP 1

$$\mathcal{L}\{y''\} - 7\mathcal{L}\{y'\} + 12\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-3) * \delta(t-5)\}$$

$$s^2\mathcal{L}\{y\} \underbrace{- s\,y(0) - y'(0)}_{0} - 7(s\mathcal{L}\{y\} - \underbrace{y(0)}_{0}) + 12\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-3)\}\mathcal{L}\{\delta(t-5)\}$$

$$(s^2 - 7s + 12)\mathcal{L}\{y\} = e^{-3s}e^{-5s}$$

$$\mathcal{L}\{y\} = \frac{e^{-8s}}{(s-3)(s-4)}$$

STEP 2 PARTIAL FRACTIONS

$$\frac{1}{(s-3)(s-4)} = \frac{A}{s-3} + \frac{B}{s-4} = \frac{A(s-4) + B(s-3)}{(s-3)(s-4)} = \frac{(A+B)s + (-4A-3B)}{(s-3)(s-4)}$$

$$\begin{cases} A + B = 0 \\ -4A - 3B = 1 \end{cases} \Longrightarrow \begin{cases} B = -A \\ -4A - 3(-A) = 1 \end{cases} \Longrightarrow \begin{cases} -A = 1 \\ B = -A \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 1 \end{cases}$$

$$\boxed{y = \left(-e^{3(t-8)} + e^{4(t-8)}\right) U_8(t)}$$

STEP 3

$$\mathcal{L}\{y\} = \left(\frac{-1}{s-3} + \frac{1}{s-4}\right)e^{-8s}$$

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$$= \mathcal{L}\{-e^{3t} + e^{4t}\}e^{-8s} = \mathcal{L}\{(-e^{3(t-8)} + e^{4(t-8)})U_8(t)\}$$

5. *(10 points)* Find a series solution of

$$\begin{cases} y'' - (2x)\,y' - 2y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Note: Write your final answer in explicit form, without series

STEP 1

$$y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{N=0}^{\infty} a_N x^N$$

$$y' = a_1 + a_2(2x) + a_3(3x^2) = \sum_{N=1}^{\infty} a_N N x^{N-1}$$

$$y'' = a_2(2) + a_3(3)(2x) + a_4(4)(3x^2) + \cdots = \sum_{N=2}^{\infty} a_N (N)(N-1) x^{N-2}$$

STEP 2

$$y'' - 2x\,y' - 2y = 0$$

$$\sum_{N=2}^{\infty} a_N (N)(N-1) x^{N-2} - 2x \sum_{N=1}^{\infty} a_N N x^{N-1} - 2 \sum_{N=0}^{\infty} a_N x^N = 0$$

$M = N-2$

$N = M+2$

$N-1 = M+1$

$$\sum_{N=0}^{\infty} a_{N+2} (N+2)(N+1) x^N - \sum_{N=1}^{\infty} 2a_N N x^N - \sum_{N=0}^{\infty} 2a_N x^N = 0$$

$$2a_2 + \sum_{N=1}^{\infty} a_{N+2} (N+2)(N+1) x^N - \sum_{N=1}^{\infty} 2a_N N x^N - 2a_0 - \sum_{N=1}^{\infty} 2a_N x^N = 0$$

$$(2a_2 - 2a_0) + \sum_{N=1}^{\infty} \left[(N+2)(N+1) a_{N+2} - 2a_N N - 2a_N \right] x^N = 0$$

$$\boxed{y = e^{x^2}}$$

$$\Rightarrow \quad 2a_2 - 2a_0 = 0 \Rightarrow \underline{a_2 = a_0}$$

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$$(N+2)(N+1) a_{N+2} - (2N+2) a_N = 0$$

$$\Rightarrow \quad a_{N+2} = \frac{2\cancel{(N+1)}}{(N+2)\cancel{(N+1)}} a_N \quad \Rightarrow \quad \underline{a_{N+2} = \frac{2}{N+2} a_N}$$

STEP 3

$y(0)=1 \Rightarrow a_0=1$

$y'(0)=0 \Rightarrow a_1=0$

CASE 1

$a_0=1$

$a_2=a_0=1$

$a_4=\frac{2}{2+2}a_2=\frac{1}{2}a_2=\frac{1}{2}$

$a_6=\frac{2}{4+2}a_4=\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)$

$a_8=\frac{2}{6+2}a_6=\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)=\frac{1}{4!}$

$a_{2N}=\frac{1}{N!}$

CASE 2

$a_1=0$, $a_3=\frac{2}{1+2}a_1=0$, $a_5=0$, ..., $a_{2N+1}=0$

STEP 4

$$y=\sum_{N=0}^{\infty}a_N x^N=\sum_{N\ \text{EVEN}}a_N x^N=\sum_{N=0}^{\infty}a_{2N}x^{2N}$$

$$y=\sum_{N=0}^{\infty}\frac{1}{N!}x^{2N}=\sum_{N=0}^{\infty}\frac{1}{N!}(x^2)^N$$

$y=e^{x^2}$

-4

6. *(10 points)* Solve $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} [redacted] \\ -2 \end{bmatrix}$ where

$$A = \begin{bmatrix} 5 & 10 \\ -2 & -3 \end{bmatrix}$$

No need to draw the phase portrait

STEP 1

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & 10 \\ -2 & -3-\lambda \end{vmatrix} = (5-\lambda)(-3-\lambda) - 10(-2)$$

$$= -15 - 5\lambda + 3\lambda + \lambda^2 + 20$$

$$= \lambda^2 - 2\lambda + 5 = (\lambda-1)^2 - 1 + 5 = (\lambda-1)^2 + 4 = 0$$

$$\Rightarrow (\lambda-1)^2 = -4 \Rightarrow \lambda - 1 = \pm 2i \Rightarrow \underline{\lambda = 1 \pm 2i}$$

STEP 2

$$\text{NUL}(A-(1+2i)) = \left[\begin{array}{cc|c} 5-(1+2i) & 10 & 0 \\ -2 & -3-(1+2i) & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 4-2i & 10 & 0 \\ -2 & -4-2i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -2 & -4-2i & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{(\div -2)} \left[\begin{array}{cc|c} 1 & 2+i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x + (2+i)y = 0 \Rightarrow x = -(2+i)y$$

$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -(2+i)y \\ y \end{bmatrix} = y \begin{bmatrix} -2-i \\ 1 \end{bmatrix}$$

STEP 3

$$e^{(1+2i)t}\left(\begin{bmatrix} -2-i \\ 1 \end{bmatrix}\right) = e^t(\cos(2t) + i\sin(2t))\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i\right)$$

$$\underline{x}(t) = C_1 e^t\left(\cos(2t)\begin{bmatrix} -2 \\ 1 \end{bmatrix} - \sin(2t)\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) + C_2 e^t\left(\cos(2t)\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin(2t)\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$$

$$\boxed{\mathbf{x}(t) = e^t \begin{bmatrix} -4\cos(2t) - 18\sin(2t) \\ -2\cos(2t) + 8\sin(2t) \end{bmatrix}}$$

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STEP 4

$$\underline{X}(0) = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

$$c_1 e^0 \left(\cos(0) \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \cancel{\sin(0)}^{\,0} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) + c_2 e^0 \left(\cos(0) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \cancel{\sin(0)}^{\,0} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$$

$$= c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2c_1 - c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -2c_1 - c_2 = -4 \\ c_1 = -2 \end{cases} \Rightarrow \begin{cases} c_1 = -2 \\ c_2 = \underbrace{-2c_1}_{4} + 4 \end{cases} \Rightarrow \begin{cases} c_1 = -2 \\ c_2 = 8 \end{cases}$$

$$\underline{x}(t) = -2e^t \left(\cos(2t) \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \sin(2t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) + 8e^t \left(\cos(2t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin(2t) \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$$

$$= e^t \begin{bmatrix} 4\cos(2t) - 2\sin(2t) - 8\cos(2t) - 16\sin(2t) \\ -2\cos(2t) + 8\sin(2t) \end{bmatrix}$$

$$= e^t \begin{bmatrix} -4\cos(2t) - 18\sin(2t) \\ -2\cos(2t) + 8\sin(2t) \end{bmatrix}$$

7. *(10 points)* Use variation of parameters to find a particular solution $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where

$$A = \begin{bmatrix} 1/t & -1 \\ 1 & 1/t \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} t \\ -t \end{bmatrix}$$

Note: Simplify your final answer and write it as a single vector. Assume the general solution of the homogeneous equation is

$$\mathbf{x}_0(t) = C_1 \begin{bmatrix} t\sin(t) \\ -t\cos(t) \end{bmatrix} + C_2 \begin{bmatrix} t\cos(t) \\ t\sin(t) \end{bmatrix}$$

STEP 1

$$X_p(t) = U(t) \begin{bmatrix} t\sin(t) \\ -t\cos(t) \end{bmatrix} + V(t) \begin{bmatrix} t\cos(t) \\ t\sin(t) \end{bmatrix}$$

$$\begin{bmatrix} t\sin(t) & t\cos(t) \\ -t\cos(t) & t\sin(t) \end{bmatrix} \begin{bmatrix} U'(t) \\ V'(t) \end{bmatrix} = \begin{bmatrix} t \\ -t \end{bmatrix}$$

STEP 2

DENOMINATOR

$$\begin{vmatrix} t\sin(t) & t\cos(t) \\ -t\cos(t) & t\sin(t) \end{vmatrix} = t^2\sin^2(t) + t^2\cos^2(t) = t^2$$

$$U'(t) = \frac{\begin{vmatrix} t & t\cos(t) \\ -t & t\sin(t) \end{vmatrix}}{t^2} = \frac{t^2\sin(t) + t^2\cos(t)}{t^2} = \sin(t) + \cos(t)$$

$$V'(t) = \begin{vmatrix} t\sin(t) & t \\ -t\cos(t) & -t \end{vmatrix} / t^2 = \frac{-t^2\sin(t) + t^2\cos(t)}{t^2} = -\sin(t) + \cos(t)$$

$\mathbf{x_p}(t) =$ $\begin{bmatrix} t \\ t \end{bmatrix}$

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STEP 3

$$U(t) = \int \sin(t) + \cos(t)\,dt$$

$$= -\cos(t) + \sin(t)$$

$$V(t) = \int -\sin(t) + \cos(t)\,dt = \cos(t) + \sin(t)$$

STEP 4

$$\underline{x}_p(t) = (-\cos(t) + \sin(t)) \begin{bmatrix} t\sin(t) \\ -t\cos(t) \end{bmatrix} + (\cos(t) + \sin(t)) \begin{bmatrix} t\cos(t) \\ t\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} -t\cancel{\cos(t)\sin(t)} + t\sin^2(t) + t\cos^2(t) + t\cancel{\cos(t)\sin(t)} \\ t\cos^2(t) - t\cancel{\sin(t)\cos(t)} + t\cancel{\cos(t)\sin(t)} + t\sin^2(t) \end{bmatrix}$$

$$= \begin{bmatrix} t \\ t \end{bmatrix}$$