SOLUTIONS

1. *(10 points)* Solve the following equation and write your answer in explicit form

$$\begin{cases} t^4y' + 3t^3y = t\cos(t) \\ y\left(\frac{\pi}{2}\right) = 0 \end{cases}$$

STEP 1

STANDARD FORM

$$y' + \left(\frac{3t^3}{t^4}\right)y = \frac{t\cos(t)}{t^4} \Rightarrow y' + \left(\frac{3}{t}\right)y = \frac{\cos(t)}{t^3}$$

STEP 2

INTEGRATING FACTOR

$$e^{\int \frac{3}{t}dt} = e^{3\ln(t)} = \left(e^{\ln(t)}\right)^3 = t^3$$

STEP 3

$$t^3\left(y' + \frac{3}{t}y\right) = \cancel{t^3}\frac{\cos(t)}{\cancel{t^3}}$$

$$(t^3y)' = \cos(t)$$

$$t^3y = \int \cos(t)\,dt = \sin(t) + C$$

$$y = \frac{\sin(t) + C}{t^3}$$

STEP 4

$$y\left(\frac{\pi}{2}\right) = 0 \Longrightarrow \frac{\overbrace{\sin\left(\frac{\pi}{2}\right)}^{1} + C}{\cancel{\left(\frac{\pi}{2}\right)^3}} = 0 \Rightarrow 1 + C = 0 \Rightarrow C = -1$$

$$y = \boxed{\frac{\sin(t) - 1}{t^3}}$$

☐ Work on Scratch Paper

2. *(10 points)* Solve the following and write your answer in explicit form.

$$\begin{cases} \frac{dy}{dx} = \frac{-2x-y}{x+1} \\ y(0) = 1 \end{cases}$$

STEP 1

$(x+1)\,dy = (-2x-y)\,dx$

$(2x+y)\,dx + (x+1)\,dy = 0$

STEP 2 CHECK EXACT:

$P_y = (2x+y)_y = 1$

$Q_x = (x+1)_x = 1$ ✓

STEP 3 FIND f

$f(x,y) = \int 2x+y\,dx = x^2+xy + \text{JUNK}$

$f(x,y) = \int x+1\,dy = xy+y + \text{JUNK}$

$f(x,y) = x^2+xy+y$

STEP 4 SOLUTION

$f(x,y) = C \Rightarrow x^2+xy+y = C$

STEP 5

$y(0) = 1 \Rightarrow 0^2 + 0(1) + 1 = C \Rightarrow C = 1$

$x^2+xy+y = 1$

$xy+y = 1-x^2 \Rightarrow y(x+1) = 1-x^2$

$y = 1-x$

$\Rightarrow y = \frac{1-x^2}{x+1} = \frac{(1-x)\cancel{(1+x)}}{\cancel{x+1}}$

☐ Work on Scratch Paper

3. *(10 points)* Use variation of parameters to solve

$$y''+4y'+4y=e^{-2t}$$

$$\begin{cases} \text{[redacted]} \\ y(0) = 1 \\ y'(0) = \text{[redacted]} -4 \end{cases}$$

STEP 1 HOMOGENEOUS SOLUTION

AUX $r^2+4r+4=0 \Rightarrow (r+2)^2=0 \Rightarrow r=-2$ REPEATED

$$y_o(t) = Ae^{-2t}+Bte^{-2t}$$

STEP 2 VARIATION OF PARAMETERS

$$y_p(t) = U(t)e^{-2t}+V(t)te^{-2t}$$

$$(te^{-2t})' = e^{-2t}-2te^{-2t} = (1-2t)e^{-2t}$$

$$\begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{bmatrix}\begin{bmatrix} U'(t) \\ V'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix}$$

DENOM
$$\begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = (1-2t)e^{-2t}e^{-2t} - te^{-2t}(-2)e^{-2t}$$
$$= (1-\cancel{2t})e^{-4t} + \cancel{2t}e^{-4t} = e^{-4t}$$

$$U'(t) = \begin{vmatrix} 0 & te^{-2t} \\ e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} \Big/ e^{-4t} = \frac{0-\cancel{te^{-2t}e^{-2t}}}{\cancel{e^{-4t}}} = -t$$

$$V'(t) = \begin{vmatrix} e^{-2t} & 0 \\ -2e^{-2t} & e^{-2t} \end{vmatrix} \Big/ e^{-4t} = \frac{(e^{-2t})(e^{-2t})-0}{e^{-4t}} = 1$$

$$\boxed{y = e^{-2t}\left(\frac{t^2}{2}-2t+1\right)}$$

$$U(t) = \int -t\,dt = -\frac{t^2}{2}$$

☑ Work on Scratch Paper

$$V(t) = \int 1\,dt = t$$

$$y_p(t) = \left(-\frac{t^2}{2}\right)e^{-2t} + t(te^{-2t}) = -\frac{t^2}{2}e^{-2t} + t^2e^{-2t} = \frac{t^2}{2}e^{-2t}$$

STEP 3 $y(t) = y_0(t) + y_p(t)$

$$y(t) = Ae^{-2t} + Bte^{-2t} + \frac{t^2}{2}e^{-2t}$$

STEP 4 $y(0) = 1$

$$Ae^0 + B(0)e^0 + \frac{0^2}{2}e^0 = 1$$

$\underline{A = 1}$

$$y(t) = e^{-2t} + Bte^{-2t} + \frac{t^2}{2}e^{-2t}$$

$$y'(t) = -2e^{-2t} + Be^{-2t} - 2Bte^{-2t} + \frac{2t}{2}e^{-2t} + \frac{t^2}{2}(-2e^{-2t})$$

$$y'(0) = -2 + B = -4 \Rightarrow \underline{B = -2}$$

$$y(t) = e^{-2t} - 2te^{-2t} + \frac{t^2}{2}e^{-2t} = e^{-2t}\left(\frac{t^2}{2} - 2t + 1\right)$$

4. *(10 points)* Use Laplace transforms to solve

$$\begin{cases} y'' + 4y = 3f(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \quad \text{where } f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ \sin(t-1) & \text{if } t \ge 1 \end{cases}$$

STEP 1 LAPLACE TRANSFORM

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 3\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} \underbrace{- sy(0) - y'(0)}_{0} = s^2\mathcal{L}\{y\}$$

$$3\mathcal{L}\{f\} = 3\mathcal{L}\{\sin(t-1)\,U_1(t)\} = 3e^{-s}\mathcal{L}\{\sin(t)\} = \frac{3e^{-s}}{s^2+1}$$

$$s^2\mathcal{L}\{y\} + 4\mathcal{L}\{y\} = \frac{3e^{-s}}{s^2+1}$$

$$\mathcal{L}\{y\} = \frac{3}{(s^2+1)(s^2+4)}e^{-s}$$

STEP 2 PARTIAL FRACTIONS

$$\frac{3}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} = \frac{(As+B)(s^2+4)+(Cs+D)(s^2+1)}{(s^2+1)(s^2+4)}$$

$$y = \boxed{\left(\sin(t-1) - \frac{1}{2}\sin(2(t-1))\right)U_1(t)}$$

$$= \frac{As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D}{(s^2+1)(s^2+4)}$$

☑ Work on Scratch Paper

$$= \frac{(A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)}{(s^2+1)(s^2+4)} = \frac{3}{(s^2+1)(s^2+4)}$$

$$\Longrightarrow \begin{cases} A+C=0 \\ B+D=0 \\ 4A+C=0 \\ 4B+D=3 \end{cases} \Longrightarrow \begin{cases} C=-A \\ D=-B \\ 4A-A=0 \Rightarrow \underline{A=0} \\ 4B-B=3 \Longrightarrow \underline{B=1} \end{cases} \Rightarrow \begin{cases} A=0 \\ B=1 \\ C=-A=0 \\ D=-B=-1 \end{cases}$$

$$\frac{3}{(s^2+1)(s^2+4)} = \frac{1}{s^2+1} - \frac{1}{s^2+4}$$

STEP 3

$$\mathcal{L}\{y\} = \left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) e^{-s}$$

$$= \mathcal{L}\left\{ \sin(t) - \frac{1}{2}\sin(2t)\right\} e^{-s}$$

$$= \mathcal{L}\left\{ \left[\sin(t-1) - \frac{1}{2}\sin(2(t-1))\right] U_1(t)\right\}$$

$$y = \left(\sin(t-1) - \frac{1}{2}\sin(2(t-1))\right) U_1(t)$$

5. *(10 points)* Find a series solution of

$$\begin{cases} y'' + (2x)\,y' + 2y = 0 \\ y(0) = \text{[redacted]}\ 1 \\ y'(0) = 0 \end{cases}$$

Note: Write your final answer in explicit form, without series

STEP 1

$$y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{N=0}^{\infty} a_N x^N$$

$$y' = a_1 + a_2(2x) + a_3(3x^2) + \cdots = \sum_{N=1}^{\infty} a_N N x^{N-1}$$

$$y'' = a_2(2) + a_3(3)(2x) + a_4(4)(3x^2) + \cdots = \sum_{N=2}^{\infty} a_N (N)(N-1) x^{N-2}$$

STEP 2

$$y'' + 2xy' + 2y = 0$$

$$\sum_{N=2}^{\infty} a_N (N)(N-1) x^{N-2} + 2x \sum_{N=1}^{\infty} a_N N x^{N-1} + 2\sum_{N=0}^{\infty} a_N x^N = 0$$

$M = N-2$

$N = M+2$

$N-1 = M+1$

$$\sum_{N=0}^{\infty} a_{N+2} (N+2)(N+1) x^N + \sum_{N=1}^{\infty} 2a_N N x^N + \sum_{N=0}^{\infty} 2a_N x^N = 0$$

$$2a_2 + \sum_{N=1}^{\infty} a_{N+2} (N+2)(N+1) x^N + \sum_{N=1}^{\infty} 2a_N N x^N + 2a_0 + \sum_{N=1}^{\infty} 2a_N x^N = 0$$

$$(2a_2 + 2a_0) + \sum_{N=1}^{\infty} \left[(N+2)(N+1) a_{N+2} + 2a_N N + 2a_N \right] x^N = 0$$

$$\boxed{y = e^{-x^2}}$$

$\Rightarrow \quad 2a_2 + 2a_0 = 0 \quad \Rightarrow \quad \underline{a_2 = -a_0}$

☑ Work on Scratch Paper

$(N+2)(N+1)\, a_{N+2} + (2N+2)\, a_N = 0$

$$\Rightarrow \quad a_{N+2} = \frac{-2\cancel{(N+1)}}{(N+2)\cancel{(N+1)}} a_N \quad \Rightarrow \quad a_{N+2} = \frac{-2}{N+2} a_N$$

STEP 3 $\quad y(0) = 1 \Rightarrow \underline{a_0 = 1}$

$y'(0) = 0 \Rightarrow \underline{a_1 = 0}$

<u>CASE 1</u> $\quad a_0 = 1$

$$a_2 = -a_0 = -1$$

$$a_4 = \frac{2}{2+2} a_2 = \left(-\frac{1}{2}\right)(-1)$$

$$a_6 = \frac{-2}{4+2} a_4 = \left(-\frac{1}{3}\right)\left(-\frac{1}{2}\right)(-1)$$

$$a_8 = \frac{-2}{6+2} a_6 = \left(-\frac{1}{4}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{2}\right)(-1) = \frac{(-1)^4}{4!}$$

$$\underline{a_{2N} = \frac{(-1)^N}{N!}}$$

<u>CASE 2</u> $\quad a_1 = 0 \qquad a_3 = \frac{2}{1+2} a_1 = 0, \quad a_5 = 0, \ldots, \quad \underline{a_{2N+1} = 0}$

STEP 4

$$y = \sum_{N=0}^{\infty} a_N x^N = \sum_{N \text{ EVEN}} a_N x^N = \sum_{N=0}^{\infty} a_{2N} x^{2N}$$

$$y = \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} x^{2N} = \sum_{N=0}^{\infty} \frac{(-x^2)^N}{N!}$$

$$y = e^{-x^2}$$

6. *(10 points)* Solve $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$ where

$$A = \begin{bmatrix} -2 & 4 \\ -6 & 8 \end{bmatrix}$$

NO need to draw the phase portrait

STEP 1 $|A - \lambda I| = \begin{vmatrix} -2-\lambda & 4 \\ -6 & 8-\lambda \end{vmatrix} = (-2-\lambda)(8-\lambda) - 4(-6)$

$= -16 + 2\lambda - 8\lambda + \lambda^2 + 24 = \lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4) = 0$

$\Rightarrow \lambda = 2$ or $\lambda = 4$

STEP 2 $\underline{\lambda = 2}$ $\text{NUL}(A - 2I) = \left[\begin{array}{cc|c} -2-2 & 4 & 0 \\ -6 & 8-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -4 & 4 & 0 \\ -6 & 6 & 0 \end{array}\right]$ $(\div -4)$ $(\div -6)$

$\rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x - y = 0 \Rightarrow x = y \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{\lambda = 4}$ $\text{NUL}(A - 4I) = \left[\begin{array}{cc|c} -2-4 & 4 & 0 \\ -6 & 8-4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -6 & 4 & 0 \\ -6 & 4 & 0 \end{array}\right]$ $(\div -2)$ $(\div -2)$

$\rightarrow \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow 3x - 2y = 0 \rightsquigarrow x = 2, y = 3 \rightsquigarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\lambda = 2 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda = 4 \rightsquigarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

STEP 3 $\underline{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

STEP 4 $\underline{x}(0) = c_1 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$

$\boxed{\mathbf{x}(t) = 3e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$

$(x-1)$ $\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 1 & 3 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \end{array}\right] (x-2) \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$

☐ Work on Scratch Paper

$\underline{x}(t) = 3e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

7. *(10 points)* Use undetermined coefficients to find a particular solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} -7t - 1 \\ -8t - 3 \end{bmatrix}$$

Note: Simplify your final answer and write it as a single vector.

STEP 1 $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 2^2 = 0$

$\Rightarrow (1-\lambda) = \pm 2 \Rightarrow \lambda = 1 \pm 2 \Rightarrow \underline{\lambda = -1}$ OR $\underline{\lambda = 3}$

STEP 2 UNDETERMINED COEFFICIENTS

$\underline{x_p}(t) = \begin{bmatrix} A \\ B \end{bmatrix} t + \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} At + C \\ Bt + D \end{bmatrix}$

$\underline{x_p}' = A\underline{x_p} + \underline{f}$

$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} At + C \\ Bt + D \end{bmatrix} + \begin{bmatrix} -7t - 1 \\ -8t - 3 \end{bmatrix}$

$\begin{bmatrix} 0t + A \\ 0t + B \end{bmatrix} = \begin{bmatrix} At + C + 2Bt + 2D - 7t - 1 \\ 2At + 2C + Bt + D - 8t - 3 \end{bmatrix}$

t-TERMS $\begin{cases} 0 = A + 2B - 7 \\ 0 = 2A + B - 8 \end{cases}$ $\rightsquigarrow A = -2B + 7$ $\rightsquigarrow 0 = 2(-2B + 7) + B - 8$

$0 = -4B + 14 + B - 8$

$3B = 6 \Rightarrow \underline{B = 2}$

$A = -2(2) + 7 \Rightarrow \underline{A = 3}$

CONSTANT TERMS

$\mathbf{x_p}(t) = \begin{bmatrix} 3t + 2 \\ 2t + 1 \end{bmatrix}$

$\begin{cases} A = C + 2D - 1 \\ B = 2C + D - 3 \end{cases} \Rightarrow \begin{cases} 3 = C + 2D - 1 \\ 2 = 2C + D - 3 \end{cases}$

☐ Work on Scratch Paper

$\Rightarrow \begin{cases} C = 4 - 2D \rightsquigarrow C = 4 - 2(1) \rightsquigarrow \underline{C = 2} \\ 2 = 2(4 - 2D) + D - 3 \rightsquigarrow 5 = 8 - 4D + D \rightsquigarrow -3D = -3 \rightsquigarrow \underline{D = 1} \end{cases}$