MATH 308 – MOCK FINAL EXAM – SOLUTIONS

1. **Separation of Variables**

STEP 1: Cross-multiplying, we get

$$\begin{aligned} 2\ln(y)dy =& xy\sqrt{1+x^2}dx \\ \left(\frac{2\ln(y)}{y}\right)dy =& x\sqrt{1+x^2}dx \\ \int\left(\frac{2\ln(y)}{y}\right)dy =& \int x\sqrt{1+x^2}dx \end{aligned}$$

Note: Here we divided by y so you might need to worry about $y=0$ as a hidden solution, but $y=0$ doesn't satisfy $y(0)=1$.

STEP 2: Integrate

For the left-side, use $u=\ln y$ then $du=\frac{1}{y}dy$

$$\int\frac{2\ln(y)}{y}dy=\int 2udu=u^2=(\ln(y))^2$$

For the right side,

$$\int x\sqrt{1+x^2}dx=\int x\left(1+x^2\right)^{\frac{1}{2}}dx=\left(\frac{1}{2}\right)\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}=\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}$$

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Here we used $u = 1 + x^2$

STEP 3:

$$(\ln(y))^2 = \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} + C$$

STEP 4: Initial Condition: $y(0) = 1$ and so

$$\begin{aligned}(\ln(1))^2 =& \frac{1}{3}\left(1+0^2\right)^{\frac{3}{2}} + C\\ 0 =& \frac{1}{3}(1)^{\frac{3}{2}} + C\\ 0 =& \frac{1}{3} + C\\ C =& -\frac{1}{3}\end{aligned}$$

STEP 5: Solution

$$\begin{aligned}(\ln(y))^2 =& \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} - \frac{1}{3}\\ (\ln(y))^2 =& \frac{1}{3}\left(\left(1+x^2\right)^{\frac{3}{2}} - 1\right)\\ \ln(y) =& \pm\sqrt{\frac{1}{3}\left(\left(1+x^2\right)^{\frac{3}{2}} - 1\right)}\end{aligned}$$

STEP 6: Answer:

$$y = e^{\pm\sqrt{\frac{1}{3}\left(\left(1+x^2\right)^{\frac{3}{2}} - 1\right)}}$$

2. **STEP 1:**

$$\begin{aligned}
\frac{dy}{dt} &=(3y)\left(1-\frac{y}{4}\right)\\
dy &=(3y)\left(1-\frac{y}{4}\right)dt\\
\frac{dy}{y\left(1-\frac{y}{4}\right)} &=3dt\\
\int \frac{dy}{y\left(\frac{4-y}{4}\right)} &= \int 3dt\\
\int \frac{4dy}{y\left(4-y\right)} &=3t+C
\end{aligned}$$

STEP 2:

$$\frac{4}{y\left(4-y\right)} = \frac{1}{y}+\frac{1}{4-y}$$

STEP 3:

$$\begin{aligned}
\int \frac{4dy}{y\left(4-y\right)} &=3t+C\\
\int \frac{1}{y}+\frac{1}{4-y}dy &=3t+C\\
\ln|y|-\ln|4-y| &=3t+C\\
\ln\left|\frac{y}{4-y}\right| &=3t+C\\
\left|\frac{y}{4-y}\right| &=e^{3t+C}\\
\frac{y}{4-y} &=\underbrace{\pm e^{C}}_{C}e^{3t}=Ce^{3t}
\end{aligned}$$

STEP 4:

$$\begin{aligned}
\frac{4-y}{y} &= \underbrace{\frac{1}{C}}_{C} e^{-3t} \\
\frac{4}{y} - 1 &= Ce^{-3t} \\
\frac{4}{y} &= 1 + Ce^{-3t} \\
\frac{1}{y} &= \frac{1 + Ce^{-3t}}{4} \\
y &= \frac{4}{1 + Ce^{-3t}}
\end{aligned}$$

STEP 5: Finally, to find C, use $y(0) = \frac{1}{2}$

$$\begin{aligned}
y(0) &= \frac{1}{2} \\
\frac{4}{1 + Ce^{0}} &= \frac{1}{2} \\
\frac{1}{1 + C} &= \frac{1}{2(4)} = \frac{1}{8} \\
1 + C &= 8 \\
C &= 7
\end{aligned}$$

STEP 6:

$$y = \frac{4}{1 + 7e^{-3t}}$$

3. **STEP 1: Homogeneous Solution**

Aux: $r^2 + 3r + 2 = 0 \Rightarrow (r+1)(r+2) = 0 \Rightarrow r = -1$ or $r = -2$

$$y_0 = Ae^{-t} + Be^{-2t}$$

STEP 2: Var of Par

$$y_p(t) = u(t)e^{-t} + v(t)e^{-2t}$$

$$\begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \sin(e^t) \end{bmatrix}$$

Denominator:

$$\begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = \left(e^{-t}\right)\left(-2e^{-2t}\right) - \left(e^{-2t}\right)\left(-e^{-t}\right) = -e^{-3t}$$

Using Cramer's rule, we get

$$u'(t) = \frac{\begin{vmatrix} 0 & e^{-2t} \\ \sin(e^t) & -2e^{-2t} \end{vmatrix}}{-e^{-3t}} = \frac{-e^{-2t}\sin(e^t)}{-e^{-3t}} = e^{-2t}e^{3t}\sin\left(e^t\right) = e^t\sin\left(e^t\right)$$

$$v'(t) = \frac{\begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & \sin(e^t) \end{vmatrix}}{-e^{-3t}} = -e^{3t}e^{-t}\sin\left(e^t\right) = -e^{2t}\sin\left(e^t\right)$$

$$u(t) = \int e^t \sin(e^t)dt = \int \sin(u)du = -\cos(u) = -\cos\left(e^t\right)$$

Here we used a $u-$sub $u = e^t$

$$\begin{aligned}
v(t) &= \int -e^{2t} \sin\left(e^t\right) dt \\
&= \int -e^t \sin\left(e^t\right) e^t dt \\
&= \int -u \sin(u) du \qquad u = e^t \\
&\overset{\text{IBP}}{=} u\cos(u) - \int \cos(u) du \\
&= u\cos(u) - \sin(u) \\
&= e^t \cos\left(e^t\right) - \sin\left(e^t\right)
\end{aligned}$$

$$\begin{aligned}
y_p &= u(t)e^{-t} + v(t)e^{-2t} \\
&= \left(-\cos\left(e^t\right)\right) e^{-t} + \left(e^t \cos\left(e^t\right) - \sin\left(e^t\right)\right) e^{-2t} \\
&= \cancel{-e^{-t} \cos\left(e^t\right)} + \cancel{e^{-t} \cos\left(e^t\right)} - e^{-2t} \sin\left(e^t\right) \\
&= -e^{-2t} \sin\left(e^t\right)
\end{aligned}$$

$$y = y_0 + y_p = Ae^{-t} + Be^{-2t} - e^{-2t} \sin\left(e^t\right)$$

4. **STEP 1:** Take Laplace Transforms

$$\begin{aligned}
\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} =& \mathcal{L}\left\{\delta(t-3) \star \left(5e^t\right)\right\} \\
\left(s^2\mathcal{L}\{y\} - \underbrace{sy(0) - y'(0)}_{0}\right) + 4\mathcal{L}\{y\} =& \mathcal{L}\{\delta(t-3)\}\,\mathcal{L}\left\{5e^t\right\} \\
\left(s^2+4\right)\mathcal{L}\{y\} =& e^{-3s}\left(\frac{5}{s-1}\right) \\
\mathcal{L}\{y\} = \frac{5}{(s-1)(s^2+4)}\left(e^{-3s}\right)&
\end{aligned}$$

STEP 3: Partial Fractions

$$\begin{aligned}
\frac{5}{(s-1)(s^2+4)} =& \frac{A}{s-1} + \frac{Bs+C}{s^2+4} \\
=& \frac{A(s^2+4) + (Bs+C)(s-1)}{(s-1)(s^2+4)} \\
=& \frac{As^2 + 4A + Bs^2 - Bs + Cs - C}{(s-1)(s^2+4)} \\
=& \frac{(A+B)s^2 + (-B+C)s + (4A-C)}{(s-1)(s^2+4)} \\
=& \frac{0s^2 + 0s + 5}{(s-1)(s^2+4)}
\end{aligned}$$

$$\begin{cases} A+B =& 0 \\ -B+C =& 0 \\ 4A-C =& 5 \end{cases}$$

The first equation becomes $A = -B$ and the second equation $C = B$ and therefore the third equation becomes

$$4A - C = 5 \Rightarrow 4(-B) - B = 5 \Rightarrow -5B = 5 \Rightarrow B = -1$$

And using $A = -B = -(-1) = 1$ and $C = B = -1$ we get:

$$\begin{cases} A = 1 \\ B = -1 \\ C = -1 \end{cases}$$

$$\frac{5}{(s-1)(s^2+4)} = \frac{1}{s-1} + \frac{-s-1}{s^2+4}$$

STEP 4:

$$\begin{aligned} \mathcal{L}\left\{y\right\} &= \left(\frac{1}{s-1} - \frac{s}{s^2+4} - \frac{1}{s^2+4}\right)\left(e^{-3s}\right) \\ &= \mathcal{L}\left\{e^t - \cos(2t) - \frac{1}{2}\sin(2t)\right\} e^{-3s} \\ &= \mathcal{L}\left\{\left(e^{t-3} - \cos(2\left(t-3\right)) - \frac{1}{2}\sin(2\left(t-3\right))\right) u_3(t)\right\} \end{aligned}$$

$$y = \left(e^{t-3} - \cos(2\left(t-3\right)) - \frac{1}{2}\sin(2\left(t-3\right))\right) u_3(t)$$

5. **STEP 1:**

$$y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

STEP 2: Plug into the ODE

$$y' = a_1 + a_2(2x) + a_3(3x^2) + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = a_2(2) + a_3(3)(2)x + a_4(4)(3)x^2 + \cdots = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

$$y'' - xy' + 2y = 0$$

$$\left(\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}\right) - x\left(\sum_{n=1}^{\infty} a_n n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$(m = n-2)$$

$$a_2(2)(1)x^0 + \sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_n n x^n + 2a_0 + \sum_{n=1}^{\infty} 2a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[a_{n+2}(n+2)(n+1) - a_n n + 2a_n\right] x^n = 0$$

This tells us that $2a_2 + 2a_0 = 0 \Rightarrow a_2 = -a_0$ and

$$\begin{aligned} a_{n+2}(n+2)(n+1) + (-n+2)a_n =& 0 \\ a_{n+2}(n+2)(n+1) =& (n-2)a_n \\ a_{n+2} =& \frac{n-2}{(n+2)(n+1)}a_n \end{aligned}$$

STEP 3: Initial Condition

$$\begin{aligned} y(0) =& 1 \Rightarrow a_0 = 1 \\ y'(0) =& 0 \Rightarrow a_1 = 0 \end{aligned}$$

Case 1: n even

$$\begin{aligned} a_0 =& 1 \\ a_2 =& -a_0 = -1 \\ a_4 =& \frac{(2-2)}{(2+2)(2+1)}a_2 = 0a_2 = 0 \\ a_6 =& \frac{(4-2)}{(4+2)(4+1)}a_4 = \frac{2}{24}(0) = 0 \end{aligned}$$

And in general $a_{2n} = 0$ except for $a_0 = 1$ and $a_2 = -1$

Case 2: n odd

$$\begin{aligned} a_1 =& 0 \\ a_3 =& \frac{(1-2)}{(1+2)(1+1)}a_1 = \frac{-1}{6}0 = 0 \\ a_5 =& 0 \end{aligned}$$

And in general $a_{2n+1} = 0$

STEP 4: Solution

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + 0x - 1x^2 + 0x^3 + \cdots = 1 - x^2$$

$$y = 1 - x^2$$

6. **STEP 1: Eigenvalues**

$$\begin{aligned}\left|A-\lambda I\right| =& \begin{vmatrix} 5-\lambda & -4 \\ 1 & 1-\lambda \end{vmatrix} \\ =&(5-\lambda)(1-\lambda)-(-4)(1) \\ =&5-5\lambda-\lambda+\lambda^2+4 \\ =&\lambda^2-6\lambda+9 \\ =&(\lambda-3)^2=0\end{aligned}$$

Which gives $\lambda = 3$ (repeated)

STEP 2:

$$\begin{aligned}e^{At} =& e^{3t}e^{(A-3I)t} \\ =& e^{3t}\left(I+(A-3I)t\right) \\ =& e^{3t}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 5-3 & -4 \\ 1 & 1-3 \end{bmatrix} t\right) \\ =& e^{3t}\begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix}\end{aligned}$$

STEP 3:

$$\mathbf{x}(t) = C_1 e^{3t}\begin{bmatrix} 1+2t \\ t \end{bmatrix} + C_2 e^{3t}\begin{bmatrix} -4t \\ 1-2t \end{bmatrix}$$

STEP 4: Initial Conditions

Either solve for C_1 and C_2 by plugging in $t = 0$, or

Fun Fact: Here $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \mathbf{x}(0) = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ (no need to solve for anything!)

STEP 5: Answer

$$\mathbf{x}(t) = 5e^{3t}\begin{bmatrix} 1+2t \\ t \end{bmatrix} + 7e^{3t}\begin{bmatrix} -4t \\ 1-2t \end{bmatrix} = e^{3t}\begin{bmatrix} 5-18t \\ 7-9t \end{bmatrix}$$

In case you're curious, here is why this works:

$$\mathbf{x}(t) = e^{At}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \Rightarrow \mathbf{x}(0) = e^{A0}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 \\ 7 \end{bmatrix} = I\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

7. **STEP 1: Homogeneous Solution**

The problem tells you that A has eigenvalues $\lambda = 2 \rightsquigarrow \begin{bmatrix}1\\1\end{bmatrix}$ and $\lambda = 4 \rightsquigarrow \begin{bmatrix}3\\4\end{bmatrix}$, and so

$$\mathbf{x_0}(t) = C_1 e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + C_2 e^{4t}\begin{bmatrix}3\\4\end{bmatrix}$$

STEP 2: Undetermined Coefficients

Here $\mathbf{f} = \begin{bmatrix}6e^{4t}\\8e^{4t}\end{bmatrix} = e^{4t}\begin{bmatrix}6\\8\end{bmatrix}$ corresponds to $\lambda = 4$, and so there is resonance, hence we guess

$$\mathbf{x_p} = \left(\begin{bmatrix}A\\B\end{bmatrix}t + \begin{bmatrix}C\\D\end{bmatrix}\right)e^{4t} = \begin{bmatrix}(At+C)e^{4t}\\(Bt+D)e^{4t}\end{bmatrix}$$

$$\begin{aligned}
\mathbf{x_p}' &= A\mathbf{x_p} + \mathbf{f}\\
\begin{bmatrix}Ae^{4t} + (At+C)4e^{4t}\\Be^{4t} + (Bt+D)4e^{4t}\end{bmatrix} &= \begin{bmatrix}-4 & 6\\-8 & 10\end{bmatrix}\begin{bmatrix}(At+C)e^{4t}\\(Bt+D)e^{4t}\end{bmatrix} + \begin{bmatrix}6e^{4t}\\8e^{4t}\end{bmatrix}\\
\begin{bmatrix}4At\cancel{e^{4t}} + (A+4C)\cancel{e^{4t}}\\4Bt\cancel{e^{4t}} + (B+4D)\cancel{e^{4t}}\end{bmatrix} &= \begin{bmatrix}(-4At-4C)\cancel{e^{4t}} + (6Bt+6D)\cancel{e^{4t}}\\(-8At-8C)\cancel{e^{4t}} + (10Bt+10D)\cancel{e^{4t}}\end{bmatrix} + \begin{bmatrix}6\cancel{e^{4t}}\\8\cancel{e^{4t}}\end{bmatrix}\\
\begin{bmatrix}4At + (A+4C)\\4Bt + (B+4D)\end{bmatrix} &= \begin{bmatrix}(-4A+6B)t + (-4C+6D+6)\\(-8A+10B)t + (-8C+10D+8)\end{bmatrix}
\end{aligned}$$

Comparing the $t-$terms, we get the system

$$\begin{cases}4A = -4A + 6B\\4B = -8A + 10B\end{cases}$$

The first equation gives us $8A = 6B \Rightarrow B = \frac{8}{6}A = \frac{4}{3}A$

The second equation also gives us $8A = 6B$ as well, so ignore it

Then for the constant terms, we get

$$\begin{cases} A + 4C = -4C + 6D + 6 \\ B + 4D = -8C + 10D + 8 \end{cases}$$

$$\begin{cases} A + 8C - 6D = 6 \\ B + 8C - 6D = 8 \end{cases}$$

Subtracting the second equation from the first, we get

$$B - A = 2 \Rightarrow \frac{4}{3}A - A = 2 \Rightarrow \frac{A}{3} = 2 \Rightarrow A = 6$$

$$B = 2 + A = 2 + 6 = 8$$

And so the system above becomes

$$\begin{cases} 6 + 8C - 6D = 6 \\ 8 + 8C - 6D = 8 \end{cases}$$

And so $8C - 6D = 0$ so $D = \frac{8}{6}C = \frac{4}{3}C$

At this point we have used up all our equations, and so we conclude that

$$\begin{cases} A = 6 \\ B = 8 \\ C = C \\ D = \frac{4}{3}C \end{cases}$$

$$\mathbf{x_p}(t) = \begin{bmatrix} (At+C)e^{4t} \\ (Bt+D)e^{4t} \end{bmatrix} = \begin{bmatrix} (6t+C)e^{4t} \\ (8t+\frac{4}{3}C)e^{4t} \end{bmatrix} = te^{4t}\begin{bmatrix} 6 \\ 8 \end{bmatrix} + Ce^{4t}\begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix}$$

Since we just need one particular solution, let $C = 0$ and so

$$\mathbf{x_p}(t) = \begin{bmatrix} 6te^{4t} \\ 8te^{4t} \end{bmatrix} = te^{4t}\begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

STEP 3: General Solution

$$\mathbf{x}(t) = \mathbf{x_0}(t) + \mathbf{x_p}(t) = C_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t}\begin{bmatrix} 3 \\ 4 \end{bmatrix} + te^{4t}\begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

STEP 4: Variation of Parameters

As before, we have

$$\mathbf{x_0}(t) = C_1\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + C_2\begin{bmatrix} 3e^{4t} \\ 4e^{4t} \end{bmatrix}$$

Variation of Parameters

$$\mathbf{x_p}(t) = u(t)\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + v(t)\begin{bmatrix} 3e^{4t} \\ 4e^{4t} \end{bmatrix}$$

$$\begin{bmatrix} e^{2t} & 3e^{4t} \\ e^{2t} & 4e^{4t} \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 6e^{4t} \\ 8e^{4t} \end{bmatrix}$$

Denominator: $\begin{vmatrix} e^{2t} & 3e^{4t} \\ e^{2t} & 4e^{4t} \end{vmatrix} = e^{2t}\left(4e^{4t}\right) - \left(3e^{4t}\right)e^{2t} = 4e^{6t} - 3e^{6t} = e^{6t}$

$$u'(t) = \frac{\begin{vmatrix} 6e^{4t} & 3e^{4t} \\ 8e^{4t} & 4e^{4t} \end{vmatrix}}{e^{6t}} = \frac{\left(6e^{4t}\right)\left(4e^{4t}\right) - \left(3e^{4t}\right)\left(8e^{4t}\right)}{e^{6t}} = 0$$

$$v'(t) = \frac{\begin{vmatrix} e^{2t} & 6e^{4t} \\ e^{2t} & 8e^{4t} \end{vmatrix}}{e^{6t}} = \frac{e^{2t}\left(8e^{4t}\right) - \left(6e^{4t}\right)e^{2t}}{e^{6t}} = \frac{2e^{6t}}{e^{6t}} = 2$$

$$u(t) = \int 0dt = 0$$
$$v(t) = \int 2dt = 2t$$

(Remember that we only need one antiderivative. 0 is *an* antiderivative of 0)

$$\mathbf{x_p}(t) = u(t)\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + v(t)\begin{bmatrix} 3e^{4t} \\ 4e^{4t} \end{bmatrix} = 0\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + 2t\begin{bmatrix} 3e^{4t} \\ 4e^{4t} \end{bmatrix} = 2te^{4t}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = te^{4t}\begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

STEP 5: General Solution

$$\mathbf{x}(t) = \mathbf{x_0}(t) + \mathbf{x_p}(t) = C_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{4t}\begin{bmatrix} 3 \\ 4 \end{bmatrix} + te^{4t}\begin{bmatrix} 6 \\ 8 \end{bmatrix}$$