1. **STEP 1: Standard Form**

$$\frac{t^2y'}{t^2} = \left(\frac{-t(t+2)}{t^2}\right) y + \frac{e^t}{t^2}$$
$$y' + \left(\frac{t+2}{t}\right) y = \frac{e^t}{t^2}$$

STEP 2: Integrating Factor

$$P = \frac{t+2}{t}$$

$$e^{\int P} = e^{\int \frac{t+2}{t} dt} = e^{\int 1+\frac{2}{t} dt} = e^{t+2\ln(t)} = e^t \left(e^{\ln(t)}\right)^2 = t^2e^t$$

STEP 3: Multiply

$$t^2e^t \left(y' + \left(\frac{t+2}{t}\right) y\right) = t^2e^t \frac{e^t}{t^2}$$
$$\left(t^2e^ty\right)' = e^{2t}$$
$$t^2e^ty = \int e^{2t}$$
$$t^2e^ty = \frac{1}{2}e^{2t} + C$$
$$y = \frac{1}{t^2e^t}\left(\frac{1}{2}e^{2t} + C\right)$$
$$y = \frac{e^t}{2t^2} + \frac{C}{t^2}e^{-t}$$

STEP 4: Answer

$$y = \frac{e^t}{2t^2} + \frac{C}{t^2}e^{-t}$$

2. **STEP 1:** Cross-Multiply

$$\begin{aligned}(4x+2y-e^x\cos(y))\,dy &= (e^x\sin(y)-2x-4y)\,dx\\(e^x\sin(y)-2x-4y)\,dx-(4x+2y-e^x\cos(y))\,dy &= 0\\(e^x\sin(y)-2x-4y)\,dx+(-4x-2y+e^x\cos(y))\,dy &= 0\end{aligned}$$

$$P=e^x\sin(y)-2x-4y \qquad Q=-4x-2y+e^x\cos(y)$$

STEP 2: Check conservative

$$\begin{aligned}P_y &= (e^x\sin(y)-2x-4y)_y = e^x\cos(y)-4\\Q_x &= (-4x-2y+e^x\cos(y))_x = -4+e^x\cos(y)\end{aligned}$$

$P_y = Q_x$ so conservative ✓

STEP 3: Find f

$$\begin{aligned}f_x &= P = e^x\sin(y)-2x-4y \Rightarrow f=\int e^x\sin(y)-2x-4y dx\\&= e^x\sin(y)-x^2-4xy+\text{ Junk}\\f_y &= Q=-4x-2y+e^x\cos(y)\Rightarrow f=\int -4x-2y+e^x\cos(y)dy\\&= -4xy-y^2+e^x\sin(y)+\text{ Junk}\end{aligned}$$

$$f(x,y)=e^x\sin(y)-x^2-y^2-4xy$$

STEP 4: General Solution

$$e^x\sin(y)-x^2-y^2-4xy=C$$

STEP 5: Initial Condition

$y(2)=0$ means $x=2$ and $y=0$

$$\begin{aligned}e^2\sin(0)-2^2-0^2-4(2)(0) &= C\\-4 &= C\\C &= -4\end{aligned}$$

STEP 6: Answer

$$e^x\sin(y)-x^2-y^2-4xy=-4$$

3. **STEP 1: Eigenvalues**

$$\begin{aligned}\left|A-\lambda I\right| =& \begin{vmatrix} 3-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} \\ =&(3-\lambda)(1-\lambda)-(-2)(1) \\ =&3-3\lambda-\lambda+\lambda^2+2 \\ =&\lambda^2-4\lambda+5 \\ =&(\lambda-2)^2-4+5 \\ =&(\lambda-2)^2+1\end{aligned}$$

$$(\lambda-2)^2=-1 \Rightarrow \lambda-2=\pm i \Rightarrow \lambda=2\pm i$$

STEP 2: $\lambda = 2+i$

$$\text{Nul }(A-(2+i)I) = \left[\begin{array}{cc|c} 3-(2+i) & -2 & 0 \\ 1 & 1-(2+i) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 1 & -1-i & 0 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $(1-i)x-2y=0$. For example $x=2$ and $y=1-i$ satisfies this, and so

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$$

STEP 3: Solution

$$\begin{aligned} e^{(2+i)t}\begin{bmatrix} 2 \\ 1-i \end{bmatrix} =& \left(e^{2t}e^{it}\right)\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\ =&e^{2t}\left(\cos(t)+i\sin(t)\right)\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\ =&e^{2t}\left(\cos(t)\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sin(t)\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\ &+ie^{2t}\left(\cos(t)\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin(t)\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)\end{aligned}$$

$$\mathbf{x}(t) = C_1e^{2t}\left(\cos(t)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \sin(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + C_2e^{2t}\left(\cos(t)\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin(t)\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

STEP 4: Initial Condition

$$\mathbf{x}(0) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$C_1 e^0 \left(\cos(0) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \sin(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + C_2 e^0 \left(\cos(0) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin(0) \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2C_1 \\ C_1 - C_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

The first line gives $2C_1 = 4 \Rightarrow C_1 = 2$ and the second one

$$C_1 - C_2 = -2 \Rightarrow 2 - C_2 = -2 \Rightarrow C_2 = 4$$

$$\begin{cases} C_1 = 2 \\ C_2 = 4 \end{cases}$$

STEP 5: Solution

$$\begin{aligned} \mathbf{x}(t) &= 2e^{2t} \left(\cos(t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \sin(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + 4e^{2t} \left(\cos(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin(t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \\ &= e^{2t} \begin{bmatrix} 4\cos(t) + 8\sin(t) \\ 2\cos(t) + 2\sin(t) - 4\cos(t) + 4\sin(t) \end{bmatrix} \\ &= e^{2t} \begin{bmatrix} 4\cos(t) + 8\sin(t) \\ -2\cos(t) + 6\sin(t) \end{bmatrix} \end{aligned}$$

4. **STEP 1: Eigenvalues**

$$\begin{aligned}
\left|A-\lambda I\right| = &\begin{vmatrix} -1-\lambda & 4 \\ -9 & 11-\lambda \end{vmatrix} \\
=&(-1-\lambda)(11-\lambda) - 4(-9) \\
=& -11+\lambda-11\lambda+\lambda^2+36 \\
=&\lambda^2-10\lambda+25 \\
=& (\lambda-5)^2
\end{aligned}$$

Which gives $\lambda = 5$ (repeated)

STEP 2:

$$\begin{aligned}
e^{At} =& e^{5t}e^{(A-5I)t} \\
=& e^{5t}\left(I+(A-5I)t\right) \\
=& e^{5t}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1-5 & 4 \\ -9 & 11-5 \end{bmatrix} t\right) \\
=& e^{5t}\begin{bmatrix} 1-6t & 4t \\ -9t & 1+6t \end{bmatrix}
\end{aligned}$$

STEP 3:

$$\mathbf{x}(t) = C_1e^{5t}\begin{bmatrix} 1-6t \\ -9t \end{bmatrix} + C_2e^{5t}\begin{bmatrix} 4t \\ 1+6t \end{bmatrix}$$

STEP 4: Initial Conditions (can skip this part)

$$\mathbf{x}(0) = C_1e^0\begin{bmatrix} 1-0 \\ 0 \end{bmatrix} + C_2e^0\begin{bmatrix} 0 \\ 1+0 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{cases} C_1 =1 \\ C_2 =2 \end{cases}$$

STEP 5: Solution

$$\mathbf{x}(t) = 1e^{5t}\begin{bmatrix} 1-6t \\ -9t \end{bmatrix} + 2e^{5t}\begin{bmatrix} 4t \\ 1+6t \end{bmatrix} = e^{5t}\begin{bmatrix} 1-6t+8t \\ -9t++2+12t \end{bmatrix} = e^{5t}\begin{bmatrix} 2t+1 \\ 3t+2 \end{bmatrix}$$

5. **Justification** (optional)

Tank 1:

$$Q_1'(t) = \underbrace{8 \times \left(\frac{1}{2}\right) + 6 \times \left(\frac{Q_2}{3}\right)}_{\text{In}} - \underbrace{6 \times \left(\frac{Q_1}{1}\right)}_{\text{Out}} = -6Q_1(t) + 2Q_2(t) + 4$$

Tank 2:

$$Q_2'(t) = \underbrace{6 \times \left(\frac{1}{3}\right) + 6 \times \left(\frac{Q_3}{6}\right)}_{\text{In}} - \underbrace{6 \times \left(\frac{Q_2}{3}\right)}_{\text{Out}} = -2Q_2(t) + Q_3(t) + 2$$

Tank 3:

$$Q_3'(t) = \underbrace{1 \times 2 + 6 \times \left(\frac{Q_1}{1}\right)}_{\text{In}} - \underbrace{6 \times \left(\frac{Q_3}{6}\right)}_{\text{Out}} = 6Q_1(t) - Q_3(t) + 2$$

System:

$$\begin{cases} Q_1'(t) = -6Q_1(t) + 2Q_2(t) + 4 \\ Q_2'(t) = -2Q_2(t) + Q_3(t) + 2 \\ Q_3'(t) = 6Q_1(t) - Q_3(t) + 2 \end{cases}$$

This is of the form $\mathbf{Q}'(t) = A\mathbf{Q}(t) + \mathbf{b}$ where

$$A = \begin{bmatrix} -6 & 2 & 0 \\ 0 & -2 & 1 \\ 6 & 0 & -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$