1. *(10 points)* Solve the following differential equation:

$$\begin{cases} y'' - 3y' - 4y = 0 \\ y(0) = -5 \\ y'(0) = 15 \end{cases}$$

1) $\underline{\text{AUX}}$ $r^2 - 3r - 4 = 0 \Rightarrow (r-4)(r+1) = 0 \Rightarrow \underline{r = 4}$ or $\underline{r = -1}$

2) $y = Ae^{4t} + Be^{-t}$

3) $\underline{y(0) = -5} \Rightarrow Ae^{0} + Be^{0} = -5 \Rightarrow A + B = -5 \Rightarrow \underline{B = -5 - A}$

$y' = 4Ae^{4t} - Be^{-t}$

4) $\underline{y'(0) = 15} \Rightarrow 4A - B = 15 \Rightarrow 4A - (-5 - A) = 15$

$\Rightarrow 4A + 5 + A = 15$

$\Rightarrow 5A = 10$

$\Rightarrow \underline{A = 2}$

$B = -5 - A = -5 - 2 = -7 \Rightarrow \underline{B = -7}$

$y = \boxed{2e^{4t} - 7e^{-t}}$

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2. *(10 points)* Solve the following differential equation

$$\begin{cases} t^2y' - y = 2t^3 e^{\frac{-1}{t}} \\ y(1) = 0 \end{cases}$$

1) $\Rightarrow y' - \left(\frac{1}{t^2}\right) y = \frac{2t^3 e^{-\frac{1}{t}}}{t^2} \Rightarrow y' - \left(\frac{1}{t^2}\right) y = 2te^{-\frac{1}{t}}$

2) $p = -\frac{1}{t^2} \Rightarrow e^{\int p} = e^{\int -\frac{1}{t^2}dt} = e^{\frac{1}{t}}$

3) $e^{\frac{1}{t}}\left(y' - \frac{1}{t^2}y\right) = \cancel{e^{\frac{1}{t}}}\, 2t\, \cancel{e^{-\frac{1}{t}}}$

$\left(e^{\frac{1}{t}} y\right)' = 2t$

$e^{\frac{1}{t}} y = t^2 + C$

$y = e^{-\frac{1}{t}}(t^2) + Ce^{-\frac{1}{t}}$

4) $y(1) = 0 \Rightarrow e^{-1}(1) + Ce^{-1} = 0 \Rightarrow C\cancel{e^{-1}} = -\cancel{e^{-1}}$

$\Rightarrow \underline{C = -1}$

$y = \boxed{t^2 e^{-\frac{1}{t}} - e^{-\frac{1}{t}}}$

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3. *(10 points)* Solve the following differential equation

$$\begin{cases} x + 3y^2\sqrt{x^2+1}\left(\dfrac{dy}{dx}\right) = 0 \\ y(0) = \text{[redacted]}\ 2 \end{cases}$$

1) $3y^2\sqrt{x^2+1}\,\dfrac{dy}{dx} = -x$

$$3y^2\frac{dy}{dx} = \frac{-x}{\sqrt{x^2+1}}$$

$$\int 3y^2\,dy = \int -\frac{x}{\sqrt{x^2+1}}\,dx$$

2) $\int 3y^2\,dy = y^3$

$$\int -\frac{x}{\sqrt{x^2+1}}\,dx \underset{\uparrow}{=} \int -\frac{1}{2\sqrt{u}}\,du = -\sqrt{u} = -\sqrt{x^2+1} + C$$

$u = x^2+1$

$du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$

$$\boxed{y = \left(-\sqrt{x^2+1} + 9\right)^{\frac{1}{3}}}$$

3) $y^3 = -\sqrt{x^2+1} + C$

$$y = \left(-\sqrt{x^2+1} + C\right)^{\frac{1}{3}}$$

4) $y(0) = 2$

$(-\sqrt{1} + C)^{\frac{1}{3}} = 2$

$(-1 + C) = 2^3 = 8$

$\underline{C = 9}$

4. *(10 points)* Solve the following differential equation

$$\begin{cases} \dfrac{dy}{dx} = -\left(\dfrac{2xy + y^2 \text{[redacted]}}{x^2 + 2xy}\right) \\ y(2) = 1 \end{cases}$$

1) $(x^2 + 2xy)\, dy = -(2xy + y^2 + 1)\, dx$

$\underbrace{(2xy + y^2 + 1)}_{P} dx + (\underbrace{x^2 + 2xy}_{Q})\, dy = 0$

2) $P_y = (2xy + y^2)$

$Q_x = (x^2 + 2xy)_x = 2x + 2y$ ✓

3) <u>FIND f</u> $f_x = P \Rightarrow f = \int P\, dx = \int 2xy + y^2\, dx = x^2y + y^2x + \text{JUNK}$

$f_y = Q \Rightarrow f = \int Q\, dy = \int x^2 + 2xy\, dy = x^2y + xy^2 + \text{JUNK}$

$f = x^2y + xy^2$

| [redacted] | $x^2y + xy^2 = 6$ |
|---|---|

4) <u>SOLUTION</u> $x^2y + xy^2 = C$ ☐ Work on Scratch Paper

5) $y(2) = 1 \Rightarrow x = 2$ AND $y = 1$

$(2^2)(1) + (1^2)(2) = C$

$C = 4 + 2 = 6$

6) $x^2y + xy^2 = 6$

5. *(10 points)* Solve the following differential equation

$$\begin{cases} y' = 2y\left(1 - \frac{y}{4}\right) \\ y(0) = 2 \end{cases}$$

1) $\frac{dy}{dt} = 2y\left(1 - \frac{y}{4}\right)$

$\int \frac{dy}{y\left(1-\frac{y}{4}\right)} = \int 2\,dt \Rightarrow \int \frac{4\,dy}{y(4-y)} = 2t + C$

$\Rightarrow \int \frac{1}{y} + \frac{1}{4-y}\,dy = 2t + C$

$\Rightarrow \ln|y| - \ln|4-y| = 2t + c$

$\Rightarrow \ln\left|\frac{y}{4-y}\right| = 2t + C$

$\Rightarrow \frac{y}{4-y} = Ce^{2t}$

$$\boxed{y = \quad 4/(1+e^{-2t})}$$

2) $\frac{4-y}{y} = Ce^{-2t}$

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$\frac{4}{y} - 1 = Ce^{-2t}$

$\frac{4}{y} = 1 + Ce^{-2t}$

$\frac{y}{4} = \frac{1}{1+Ce^{-2t}} \Rightarrow y = \frac{4}{1+Ce^{-2t}}$

3) $y(0)=2 \Rightarrow \frac{4}{1+C} = 2$

$\Rightarrow \frac{1}{1+C} = \frac{2}{4} = \frac{1}{2} \Rightarrow 1+C=2$

$\Rightarrow C=1$

6. *(10 points)* Consider the following ODE:

$$t^2y'' - 4ty' + 4y = 0 \implies y'' - \frac{4}{t}y' + \frac{4}{t^2}y = 0$$

(a) Use Abel's Formula to find the Wronskian $W(t)$

(b) Suppose one solution is $f(t) = t$ (do not check). Find another solution $g(t)$ that is not a multiple of $f(t)$

(c) Find the general solution $y$ of the ODE above

(a) $p = -\frac{4}{t}$ $\quad W(t) = Ce^{-\int p} = Ce^{\int \frac{4}{t}} = Ce^{4\ln(t)} = Ct^4$

$\leadsto \underline{W(t) = t^4}$

(b) $W(t) = \begin{vmatrix} t & g \\ 1 & g' \end{vmatrix} = t^4$

$\implies tg' - g = t^4$

$\implies g' - \left(\frac{1}{t}\right)g = t^3$ $\quad p = -\frac{1}{t} \implies e^{\int p} = e^{-\int \frac{1}{t}} = e^{-\ln(t)} = \frac{1}{t}$

| | |
|---|---|
| $W(t) =$ | $t^4$ |
| $g(t) =$ | $t^4/3$ |
| $y =$ | $At + Bt^4$ |

$\implies \left(\frac{1}{t}\right)g' - \left(\frac{1}{t^2}\right)g = \left(\frac{1}{t}\right)t^3$

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$\implies \left(\frac{1}{t}g\right)' = t^2$

$\implies \frac{1}{t}g = \int t^2$

$\frac{1}{t}g = \frac{t^3}{3} \implies$

$\implies \underline{g = \frac{t^4}{3}}$

(c) $Af(t) + Bg(t) = At + B(t^4/3) = \underline{At + Bt^4}$