MATH 308 – MOCK MIDTERM 1 – SOLUTIONS

1. **STEP 1:**

$$\begin{aligned}
\frac{dy}{dt} =& (2y)\left(1-\frac{y}{6}\right) \\
dy =& (2y)\left(1-\frac{y}{6}\right)dt \\
\frac{dy}{y\left(1-\frac{y}{6}\right)} =& 2dt \\
\int \frac{dy}{y\left(\frac{6-y}{6}\right)} =& \int 2dt \\
\int \frac{6dy}{y\left(6-y\right)} =& 2t + C
\end{aligned}$$

STEP 2:

$$\frac{6}{y\left(6-y\right)} = \frac{1}{y} + \frac{1}{6-y}$$

STEP 3:

Date: Monday, February 21, 2022.

$$\begin{aligned}
\int \frac{6dy}{y\,(6-y)} =& 2t + C \\
\int \frac{1}{y} + \frac{1}{6-y} dy =& 2t + C \\
\ln|y| - \ln|6-y| =& 2t + C \\
\ln\left|\frac{y}{6-y}\right| =& 2t + C \\
\left|\frac{y}{6-y}\right| =& e^{2t+C} \\
\frac{y}{6-y} =& \underbrace{\pm e^{C}}_{C} e^{2t} = Ce^{2t}
\end{aligned}$$

STEP 4:

$$\begin{aligned}
\frac{6-y}{y} =& \underbrace{\frac{1}{C}}_{C} e^{-2t} \\
\frac{6}{y} - 1 =& Ce^{-2t} \\
\frac{6}{y} =& 1 + Ce^{-2t} \\
\frac{1}{y} =& \frac{1 + Ce^{-2t}}{6} \\
y =& \frac{6}{1 + Ce^{-2t}}
\end{aligned}$$

STEP 5: Finally, to find C, use $y(0) = 3$

$$\begin{aligned} y(0) =& 3 \\ \frac{6}{1+Ce^0} =& 3 \\ \frac{1}{1+C} =& \frac{3}{6} = \frac{1}{2} \\ 1+C =& \frac{1}{\frac{1}{2}} \\ 1+C =& 2 \\ C =& 1 \end{aligned}$$

STEP 6:

$$y = \frac{6}{1+e^{-2t}}$$

2. **STEP 1:** Standard form: Divide by $2t^2$

$$y'' + \left(\frac{3t}{2t^2}\right) y' - \left(\frac{1}{2t^2}\right) y = 0$$
$$y'' + \left(\frac{3}{2t}\right) y' - \left(\frac{1}{2t^2}\right) y = 0$$

$$P(t) = \frac{3}{2t}$$

STEP 2: By Abel's Formula:

$$W(t) = Ce^{-\int P} = Ce^{-\int \frac{3}{2t}} = Ce^{-\frac{3}{2}\int \frac{1}{t}} = Ce^{-\frac{3}{2}\ln(t)} = C\left(e^{\ln(t)}\right)^{-\frac{3}{2}} = Ct^{-\frac{3}{2}}$$

$$W(t) = t^{-\frac{3}{2}}$$

STEP 3:

$$W(t) = \begin{vmatrix} \frac{1}{t} & g(t) \\ -\frac{1}{t^2} & g'(t) \end{vmatrix} = \left(\frac{1}{t}\right) g'(t) + \left(\frac{1}{t^2}\right) g(t) \overset{\text{WANT}}{=} t^{-\frac{3}{2}}$$

STEP 4:

$$\left(\frac{1}{t}\right) g' + \left(\frac{1}{t^2}\right) g = t^{-\frac{3}{2}}$$

Multiply by t (standard form)

$$g' + \frac{1}{t}g = t^{-\frac{1}{2}}$$

$$P = \frac{1}{t} \Rightarrow e^{\int P} = e^{\int \frac{1}{t}} = e^{\ln(t)} = t$$

$$\begin{aligned}
tg' + t\left(\frac{1}{t}\right)g =& tt^{-\frac{1}{2}} \\
tg' + g =& t^{\frac{1}{2}} \\
(tg)' =& t^{\frac{1}{2}} \\
tg =& \int t^{\frac{1}{2}} \\
tg =& \frac{2}{3}t^{\frac{3}{2}} \\
g =& \frac{1}{t}\left(\frac{2}{3}\right)t^{\frac{3}{2}} \\
g =& \frac{2}{3}\sqrt{t}
\end{aligned}$$

This gives $g(t) = \frac{2}{3}\sqrt{t}$

STEP 5: The general solution is

$$y = Af(t) + Bg(t) = \frac{A}{t} + B\left(\frac{2}{3}\right)\sqrt{t} = \frac{A}{t} + B\sqrt{t}$$

3. **STEP 1:** Standard Form

$$y'' + \left(\frac{\ln(t-1)}{\sqrt{9-t^2}}\right) y' + \left(\frac{\cos(t)}{\sqrt{9-t^2}}\right) y = \left(\frac{|t-4|}{\sqrt{9-t^2}}\right)$$

STEP 2: Continuous, but just the part with $t = 2$

$$\frac{\ln(t-1)}{\sqrt{9-t^2}} \Rightarrow t > 1 \text{ and } 9 - t^2 > 0 \Rightarrow (1,\infty) \text{ and } (-3,3) \Rightarrow (1,3)$$

$$\frac{\cos(t)}{\sqrt{9-t^2}} \Rightarrow (-3,3)$$

$$\frac{|t-4|}{\sqrt{9-t^2}} \Rightarrow (-3,3) \text{ Remember } |t-4| \text{ is continuous everywhere}$$

STEP 3: Find the intersection of all those intervals:

$$(1,3) \text{ and } (-3,3) \text{ and } (-3,3) \Rightarrow (1,3)$$

4. **STEP 1:** Integrating Factors

$$S' - 0.1S = 10$$

Using the integrating factor $e^{-0.1t}$, we get

$$\begin{aligned}
e^{-0.1t}S' - 0.1e^{-0.1t}S =& 10e^{-0.1t} \\
\left(e^{-0.1t}S\right)' =& 10e^{-0.1t} \\
e^{-0.1t}S =& \int 10e^{-0.1t}dt \\
=& \left(\frac{1}{-0.1}\right) 10e^{-0.1t} + C \\
e^{-0.1t}S =& -100e^{-0.1t} + C \\
S(t) =& -100 + Ce^{0.1t}
\end{aligned}$$

STEP 2: To solve for C, use:

$$\begin{aligned}
S(0) =& S_0 \\
-100 + Ce^0 =& S_0 \\
-100 + C =& S_0 \\
C =& S_0 + 100
\end{aligned}$$

STEP 3: Solution

$$S(t) = -100 + (S_0 + 100)\, e^{0.1t}$$

STEP 4:

$$\begin{aligned}
S(t) &= 3S_0 \\
-100 + (S_0 + 100)\, e^{0.1t} &= 3S_0 \\
(S_0 + 100)\, e^{0.1t} &= 3S_0 + 100 \\
e^{0.1t} &= \frac{3S_0 + 100}{S_0 + 100} \\
0.1t &= \ln\left(\frac{3S_0 + 100}{S_0 + 100}\right) \\
t &= 10 \ln\left(\frac{3S_0 + 100}{S_0 + 100}\right)
\end{aligned}$$

5. **STEP 1:** Using the hint, we first multiply by $\frac{1}{xy^3}$

$$\begin{aligned}\frac{x^2y^3}{xy^3} + \frac{x\left(1+y^2\right)}{xy^3}y' =&0\\ x + \frac{1+y^2}{y^3}\left(\frac{dy}{dx}\right) =&0\\ xdx + \frac{1+y^2}{y^3}dy =&0\end{aligned}$$

STEP 2: Check exact

$$\begin{aligned}P =&x\\ Q =&\frac{1+y^2}{y^3}\end{aligned}$$

$$\begin{aligned}P_y &= (x)_y = 0\\ Q_x &= \left(\frac{1+y^2}{y^3}\right)_x = 0\checkmark\end{aligned}$$

STEP 3: Integrate

$$f_x = P \Rightarrow f = \int P dx = \int x dx = \frac{x^2}{2} + \text{ Junk}$$

$$f_y = Q \Rightarrow f = \int Q dy$$

$$
\begin{aligned}
f &= \int \frac{1+y^2}{y^3} dy \\
&= \int \frac{1}{y^3} + \frac{y^2}{y^3} dy \\
&= \int y^{-3} + \frac{1}{y} dy \\
&= \frac{y^{-2}}{-2} + \ln|y| + \text{ Junk} \\
&= -\frac{1}{2y^2} + \ln|y| + \text{ Junk}
\end{aligned}
$$

$$f = \frac{x^2}{2} - \frac{1}{2y^2} + \ln|y|$$

STEP 4: Solution

$$\frac{x^2}{2} - \frac{1}{2y^2} + \ln|y| = C$$

6. (a) $4y'' - y = 0$

$$\textbf{Aux: } 4r^2 - 1 = 0 \Rightarrow 4r^2 = 1 \Rightarrow r^2 = \frac{1}{4} \Rightarrow r = \pm\frac{1}{2}$$

$$y = Ae^{\frac{t}{2}} + Be^{-\frac{t}{2}}$$

(b) $y'' + 4y' + 4y = 0$

$$\textbf{Aux: } r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow r = -2 \text{ (Double Root)}$$

$$y = Ae^{-2t} + Bte^{-2t}$$

(c) $4y'' + 8y' + 5y = 0$

Auxiliary Equation: $4r^2 + 8r + 5 = 0$

$$\begin{aligned} r =& \frac{-8 \pm \sqrt{8^2 - 4(4)(5)}}{2(4)} \\ =& \frac{-8 \pm \sqrt{64 - 80}}{8} \\ =& \frac{-8 \pm \sqrt{-16}}{8} \\ =& \frac{-8 \pm 4i}{8} \\ =& -1 \pm \left(\frac{1}{2}\right) i \end{aligned}$$

$$y = Ae^{-t}\cos\left(\frac{t}{2}\right) + Be^{-t}\sin\left(\frac{t}{2}\right)$$