SOLUTIONS

1. *(10 = 5 ×2 points)* Guess the form of the particular sol of

(a) $y'' + 4y = \sin(3t)$
(b) $y'' + 4y = e^{3t}\cos(2t)$
(c) $y'' + 4y = t\sin(2t)$
(d) $y'' + 4y = e^{2t}$
(e) $y'' + 4y = t^3 + 3$

Note: No justification required here

NOTE FOR THE HOMOGENEOUS SOLUTION, WE HAVE $y'' + 4y = 0$

AUX $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$y_0(t) = A\cos(2t) + B\sin(2t)$

(a) $\sin(3t) \rightsquigarrow r = 3i$ NO RESONANCE

(b) $e^{3t}\cos(2t) \rightsquigarrow r = 3+2i$ NO RESONANCE

(c) $\sin(2t) \rightsquigarrow r = 2i$ RESONANCE

(d) $e^{2t} \rightsquigarrow r = 2$ NO RESONANCE

(e) NO RESONANCE AND $t^3 + 2$ IS A CUBIC POLYNOMIAL

(a)	$y_p = A\cos(3t) + B\sin(3t)$
(b)	$y_p = Ae^{3t}\cos(2t) + Be^{3t}\sin(2t)$
(c)	$y_p = t(At+B)\cos(2t) + t(Ct+D)\sin(2t)$
(d)	$y_p = Ae^{2t}$
(e)	$y_p = At^3 + Bt^2 + Ct + D$

☐ Work on Scratch Paper

2. *(10 points)* Use **variation of parameters** to find a particular solution to

$$y'' - 4y' + 4y = \frac{e^{2t}}{t^2}$$

1) HOMOGENEOUS $\quad y'' - 4y' + 4y = 0 \rightsquigarrow r^2 - 4r + 4 = 0 \rightsquigarrow r = 2$

$y_0 = Ae^{2t} + Bte^{2t}$

2) VAR OF PAR $\quad y_p = U(t)e^{2t} + V(t)te^{2t}$

$$\begin{bmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (1+2t)e^{2t} \end{bmatrix} \begin{bmatrix} U'(t) \\ V'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ (e^{2t})/t^2 \end{bmatrix}$$

$(te^{2t})' = e^{2t} + t(2e^{2t}) = (1+2t)e^{2t}$

DENOM $\quad \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (1+2t)e^{2t} \end{vmatrix} = e^{2t}(1+2t)e^{2t} - te^{2t}(2)e^{2t} = (1+2t-2t)e^{4t} = e^{4t}$

$$U'(t) = \frac{\begin{vmatrix} 0 & te^{2t} \\ \frac{e^{2t}}{t^2} & (1+2t)e^{2t} \end{vmatrix}}{e^{4t}} = \frac{0 - te^{2t}\left(\frac{e^{2t}}{t^2}\right)}{e^{4t}} = \frac{-e^{4t}}{te^{4t}} = -\frac{1}{t}$$

$$V'(t) = \frac{\begin{vmatrix} e^{2t} & 0 \\ 2e^{2t} & (e^{2t})/t^2 \end{vmatrix}}{e^{4t}} = \frac{e^{2t}\frac{(e^{2t})}{t^2} - 0}{e^{4t}} = \frac{\cancel{e^{4t}}}{\cancel{e^{4t}}(t^2)} = \frac{1}{t^2}$$

$U(t) = \int -\frac{1}{t} = -\ln|t| \qquad V(t) = \int \frac{1}{t^2} = -\frac{1}{t}$

$$\boxed{y_p = (-\ln|t| - 1)e^{2t}}$$

☐ Work on Scratch Paper

$y_p = U(t)e^{2t} + V(t)(te^{2t})$

$= -\ln|t|e^{2t} + \left(-\frac{1}{\cancel{t}}\right)(\cancel{t}e^{2t}) = -\ln|t|e^{2t} - e^{2t} = (-\ln|t| - 1)e^{2t}$

3. *(10 points)* Use Laplace Transforms to solve

$$\begin{cases} y'' - 3y' - 10y = 14\,\delta(t-4) \\ y(0) = 1 \\ y'(0) = -2 \end{cases}$$

1) $\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} - 10\mathcal{L}\{y\} = 14\mathcal{L}\{\delta(t-4)\}$

$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 3(s\mathcal{L}\{y\} - y(0)) - 10\mathcal{L}\{y\} = 14e^{-4s}$

$(s^2 - 3s - 10)\mathcal{L}\{y\} - s + 2 + 3(y(0)) = 14e^{-4s}$

$(s^2-3s-10)\mathcal{L}\{y\} = s - 5 + 14e^{-4s}$

$\mathcal{L}\{y\} = \dfrac{s-5}{s^2-3s-10} + \left(\dfrac{14}{s^2-3s-10}\right)e^{-4s}$

2) NOTE $\dfrac{s-5}{s^2-3s-10} = \dfrac{\cancel{s-5}}{(\cancel{s-5})(s+2)} = \dfrac{1}{s+2}$

$\dfrac{14}{s^2-3s-10} = \dfrac{14}{(s-5)(s+2)} = \dfrac{A}{s-5} + \dfrac{B}{s+2} = \dfrac{A(s+2)+B(s-5)}{(s-5)(s+2)}$

$= \dfrac{As + 2A + Bs - 5B}{(s-5)(s+2)} = \dfrac{(A+B)s + (2A-5B)}{(s-5)(s+2)} = \dfrac{14}{(s-5)(s+2)}$

$\begin{cases} A+B=0 \\ 2A-5B=14 \end{cases} \Longrightarrow \begin{cases} B=-A \\ 2A-5(-A)=14 \end{cases} \Longrightarrow \begin{cases} B=-A \\ 7A=14 \end{cases} \Longrightarrow \begin{cases} A=2 \\ B=-2 \end{cases}$

$$\boxed{y = \; e^{-2t} + \left(2e^{5(t-4)} - 2e^{-2(t-4)}\right)U_4(t)}$$

3) $\mathcal{L}\{y\} = \dfrac{1}{s+2} + \left(\dfrac{2}{s-5} - \dfrac{2}{s+2}\right)e^{-4s}$ ☐ Work on Scratch Paper

$= \mathcal{L}\{e^{-2t}\} + \mathcal{L}\{2e^{5t} - 2e^{-2t}\}e^{-4s}$

$= \mathcal{L}\{e^{-2t} + (2e^{5(t-4)} - 2e^{-2(t-4)})U_4(t)\}$

4. *(10 points)* Find a function whose Laplace transform is

$$\left(\frac{2s-3}{4s^2-12s+25}\right)\left(\frac{1}{s^2+9}\right)$$

Write your answer in terms of an integral

Hint: Do both terms separately

1) $\dfrac{2s-3}{4s^2-12s+25} = \dfrac{2s-3}{4\left(s^2-3s+\frac{25}{4}\right)} = \dfrac{2s-3}{4\left(\left(s-\frac{3}{2}\right)^2-\frac{9}{4}+\frac{25}{4}\right)}$

$= \dfrac{2s-3}{4\left[\left(s-\frac{3}{2}\right)^2+4\right]}$ $\quad U = s-\frac{3}{2}$ THEN $2s-3 = 2U$

$= \dfrac{2U}{4(U^2+4)} = \dfrac{1}{2}\left(\dfrac{U}{U^2+4}\right) = \mathcal{L}\left\{\frac{1}{2}\cos(2t)\right\}$

HENCE $\dfrac{2s-3}{4s^2-12s+25} = \mathcal{L}\left\{e^{\frac{3t}{2}}\left(\frac{1}{2}\cos(2t)\right)\right\}$

2) $\dfrac{1}{s^2+9} = \dfrac{1}{3}\left(\dfrac{3}{s^2+9}\right) = \mathcal{L}\left\{\frac{1}{3}\sin(3t)\right\}$

3) ANS $\left(e^{\frac{3t}{2}}\left(\frac{1}{2}\right)\cos(2t)\right) * \left(\frac{1}{3}\sin(3t)\right)$

Answer: $\displaystyle \int_0^t \frac{1}{6} e^{\frac{3(t-\tau)}{2}} \cos(2(t-\tau))\sin(3\tau)\,d\tau$

☐ Work on Scratch Paper

5. *(10 points)* Find a recurrence relation for the coefficients a_n in a series solution of

$$y'' + xy' + y = 0$$

Simplify your answer

Note: You do not need to find an explicit formula for the a_n

1) $y = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{N=0}^{\infty} a_N x^N$

$y' = a_1 + a_2(2x) + a_3(3x^2) + \dots = \sum_{N=1}^{\infty} a_N N x^{N-1}$

$y'' = a_2(2) + a_3(3)(2)x + a_4(4)(3)x^2 + \dots = \sum_{N=2}^{\infty} a_N (N)(N-1)x^{N-2}$

2) $y'' + xy' + y = 0$

$$\left(\sum_{N=2}^{\infty} a_N (N)(N-1) x^{N-2}\right) + x\left(\sum_{N=1}^{\infty} a_N (N) x^{N-1}\right) + \sum_{N=0}^{\infty} a_N x^N = 0$$

$M = N-2$

$$\sum_{N=2}^{\infty} a_N (N)(N-1) x^{N-2} + \sum_{N=1}^{\infty} a_N (N) x^N + \sum_{N=0}^{\infty} a_N x^N = 0$$

$$\sum_{N=0}^{\infty} a_{N+2} (N+2)(N+1) x^N + \sum_{N=1}^{\infty} a_N (N) x^N + \sum_{N=0}^{\infty} a_N x^N = 0$$

$$\underline{a_2(2)(1) + a_0} + \sum_{N=1}^{\infty} \underline{\left[a_{N+2}(N+2)(N+1) + N a_N + a_N\right]} x^N = 0$$

Answer	$a_{N+2} = -\dfrac{a_N}{N+2}$

$2a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{1}{2} a_0$

☐ Work on Scratch Paper

$(N+2)\cancel{(N+1)} a_{N+2} + \cancel{(N+1)} a_N = 0$

$a_{N+2} = -a_N / (N+2)$ (NOTICE THIS INCLUDES $a_2 = -\frac{a_0}{2}$)