MATH 308 – MOCK MIDTERM 2 – SOLUTIONS

1. **STEP 1: Homogeneous Solution**

$$\textbf{Aux: } r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2$$

$$y_0 = Ae^{2t} + Bte^{2t}$$

STEP 2: Particular Solution

The right-hand-side corresponds to $r = 2$, which coincides with the double root $r = 2$, therefore

$$y_p = t^2(At+B)e^{2t} = (At^3 + Bt^2)e^{2t}$$

$$\begin{aligned}(y_p)' &= \left(3At^2 + 2Bt\right)e^{2t} + \left(At^3 + Bt^2\right)2e^{2t} \\ &= \left(3At^2 + 2Bt + 2At^3 + 2Bt^2\right)e^{2t} \\ &= \left(2At^3 + (3A+2B)t^2 + 2Bt\right)e^{2t}\end{aligned}$$

$$\begin{aligned}(y_p)'' &= \left(6At^2 + (3A+2B)(2t) + 2B\right)e^{2t} + \left(2At^3 + (3A+2B)t^2 + 2Bt\right)\left(2e^{2t}\right) \\ &= \left(6At^2 + (6A+4B)\,t + 2B + 4At^3 + (6A+4B)\,t^2 + 4Bt\right)e^{2t} \\ &= \left(4At^3 + (12A+4B)t^2 + (6A+8B)t + 2B\right)e^{2t}\end{aligned}$$

$$\begin{aligned}(y_p)'' - 4\,(y_p)' + 4\,(y_p) &= 12te^{2t} \\ \left(4At^3 + (12A+4B)t^2 + (6A+8B)t + 2B\right)\cancel{e^{2t}} - 4\left(2At^3 + (3A+2B)t^2 + 2Bt\right)\cancel{e^{2t}} & \\ +4\left(At^3 + Bt^2\right)\cancel{e^{2t}} &= 12t\cancel{e^{2t}}\end{aligned}$$

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$$\begin{aligned} \cancel{4At^3} + (12A+4B)t^2 + (6A+8B)t + 2B - \cancel{8At^3} + (-12A-8B)t^2 & \\ -8Bt + \cancel{4At^3} + 4Bt^2 &= 12t \\ \cancel{12At^2} + \cancel{4Bt^2} + 6At + \cancel{8Bt} + 2B - \cancel{12At^2} - \cancel{8Bt^2} - \cancel{8Bt} + \cancel{4Bt^2} &= 12t \\ 6At + 2B &= 12t + 0 \end{aligned}$$

$$\begin{cases} 6A = 12 \\ 2B = 0 \end{cases}$$

Which gives $A = 2$ and $B = 0$ and therefore

$$y_p = \left(At^3 + Bt^2\right) e^{2t} = 2t^3 e^{2t}$$

STEP 3: General Solution

$$y = y_0 + y_p = Ae^{2t} + Bte^{2t} + 2t^3 e^{2t}$$

STEP 4: Initial Conditions

$$y(0) = 0 \Rightarrow Ae^0 + B(0)e^0 + 2\,(0)^3\, e^0 = 0 \Rightarrow A = 0$$

$$y = Bte^{2t} + 2t^3 e^{2t}$$

$$y' = Be^{2t} + 2Bte^{2t} + 6t^2 e^{2t} + 2t^3(2e^{2t})$$

$$y'(0) = 2 \Rightarrow Be^0 + 2B(0)e^0 + 6\,(0)^2\, e^0 + 2\,(0)^3\,(2e^0) = 2 \Rightarrow B = 2$$

$$y = 2te^{2t} + 2t^3 e^{2t}$$

2. **STEP 1: Homogeneous Solution**

$$\textbf{Aux: } r^2 + 4 = 0 \Rightarrow r = \pm 2i$$

$$y_0 = A\cos(2t) + B\sin(2t)$$

STEP 2: Var of Par

$$y_p(t) = u(t)\cos(2t) + v(t)\sin(2t)$$

$$\begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 20\cos(3t) \end{bmatrix}$$

Denominator:

$$\begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2\cos^2(2t) + 2\sin^2(2t) = 2$$

Using Cramer's rule, we get

$$u'(t) = \frac{\begin{vmatrix} 0 & \sin(2t) \\ 20\cos(3t) & 2\cos(2t) \end{vmatrix}}{2} = \frac{0 - \sin(2t)20\cos(3t)}{2} = -10\sin(2t)\cos(3t)$$

$$v'(t) = \frac{\begin{vmatrix} \cos(2t) & 0 \\ -2\sin(2t) & 20\cos(3t) \end{vmatrix}}{2} = \frac{\cos(2t)20\cos(3t) - 0}{2} = 10\cos(2t)\cos(3t)$$

$$\begin{aligned}
u(t) =& -10 \int \sin(2t)\cos(3t)dt \\
=& -\frac{10}{2} \int \sin(2t+3t) + \sin(2t-3t)dt \qquad \text{Using the hint} \\
=& -5 \int \sin(5t) + \sin(-t)dt \\
=& -5 \int \sin(5t) - \sin(t)dt \\
=& -5 \left(-\frac{1}{5}\cos(5t) + \cos(t) \right) \\
=& \cos(5t) - 5\cos(t)
\end{aligned}$$

$$\begin{aligned}
v(t) =& 10 \int \cos(2t)\cos(3t)dt \\
=& \frac{10}{2} \int \cos(2t-3t) + \cos(2t+3t)dt \\
=& 5 \int \cos(-t) + \cos(5t)dt \\
=& 5 \int \cos(t) + \cos(5t)dt \\
=& 5 \left(\sin(t) + \frac{1}{5}\sin(5t) \right) dt \\
=& 5\sin(t) + \sin(5t)
\end{aligned}$$

$$\begin{aligned}
y_p(t) =& u(t)\cos(2t) + v(t)\sin(2t) \\
=& (\cos(5t) - 5\cos(t))\cos(2t) + (5\sin(t) + \sin(5t))\sin(2t) \\
=& \cos(5t)\cos(2t) - 5\cos(t)\cos(2t) + 5\sin(t)\sin(2t) + \sin(5t)\sin(2t)
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2}\left[\cos(5t-2t)+\cos(5t+2t)-5\cos(t-2t)-5\cos(t+2t)\right.\\
&\left.-5\cos(t+2t)+5\cos(t-2t)-\cos(5t+2t)+\cos(5t-2t)\right]\\
=&\frac{1}{2}\left[\cos(3t)+\cancel{\cos(7t)}-\cancel{5\cos(-t)}-5\cos(3t)\right.\\
&\left.-5\cos(3t)+\cancel{5\cos(-t)}-\cancel{\cos(7t)}+\cos(3t)\right]\\
=&\frac{1}{2}\left(2\cos(3t)-10\cos(3t)\right)\\
=&\frac{1}{2}\left(-8\cos(3t)\right)\\
=&-4\cos(3t)
\end{aligned}
$$

$$y_p = -4\cos(3t)$$

3. **STEP 1:** First write f in terms of u_c

Start at 2 then jump down by 2 at 10 and so

$$f(t) = 2 - 2u_{10}(t)$$

Therefore we need to solve

$$\begin{cases} y'' + 3y' + 2y = 2 - 2u_{10}(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

STEP 2: Take Laplace Transforms

$$\mathcal{L}\{2 - 2u_{10}(t)\} = \frac{2}{s} - 2\frac{e^{-10s}}{s} = \frac{2}{s}\left(1 - e^{-10s}\right)$$

$$\begin{aligned}
\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= \mathcal{L}\{f(t)\} \\
\left(s^2\mathcal{L}\{y\} - \underbrace{sy(0) - y'(0)}_{0}\right) + 3\left(s\mathcal{L}\{y\} - y(0)\right) + 2\mathcal{L}\{y\} &= \frac{2}{s}\left(1 - e^{-10s}\right) \\
\left(s^2 + 3s + 2\right)\mathcal{L}\{y\} &= \frac{2}{s}\left(1 - e^{-10s}\right) \\
\mathcal{L}\{y\} = \frac{2}{s(s^2 + 3s + 2)}\left(1 - e^{-10s}\right)
\end{aligned}$$

STEP 3: Partial Fractions

$$\begin{aligned}
\frac{2}{s(s^2+3s+2)} =& \frac{2}{s(s+1)(s+2)} \\
=& \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \\
=& \frac{A(s+1)(s+2) + Bs(s+2) + Cs(s+1)}{s(s+1)(s+2)} \\
=& \frac{As^2 + 3As + 2A + Bs^2 + 2Bs + Cs^2 + Cs}{s(s+1)(s+2)} \\
=& \frac{(A+B+C)s^2 + (3A+2B+C)s + 2A}{s(s+1)(s+2)} \\
=& \frac{0s^2 + 0s + 2}{s(s+1)(s+2)}
\end{aligned}$$

$$\begin{cases} A+B+C =0 \\ 3A+2B+C =0 \\ 2A =2 \end{cases}$$

The third equation gives $A = 1$ and the first two become

$$\begin{cases} B+C =-A = -1 \\ 2B+C =-3A = -3 \end{cases}$$

Subtracting the second equation from the third, we get $B = -3+1 = -2$ and finally $C = -1 - B = -1 + 2 = 1$

$$\begin{cases} A =1 \\ B =-2 \\ C =1 \end{cases}$$

$$\frac{2}{s(s^2+3s+2)} = \frac{1}{s} + \frac{-2}{s+1} + \frac{1}{s+2}$$

STEP 4:

$$\begin{aligned}
\mathcal{L}\{y\} =& \frac{2}{s(s^2+3s+2)}\left(1-e^{-10s}\right) \\
=& \left(\frac{1}{s} + \frac{-2}{s+1} + \frac{1}{s+2}\right)\left(1-e^{-10s}\right) \\
=& \mathcal{L}\left\{1-2e^{-t}+e^{-2t}\right\}\left(1-e^{-10s}\right) \\
=& \mathcal{L}\left\{1-2e^{-t}+e^{-2t}-\left(1-2e^{-(t-10)}+e^{-2(t-10)}\right)u_{10}(t)\right\}
\end{aligned}$$

$$y = 1-2e^{-t}+e^{-2t}-\left(1-2e^{-(t-10)}+e^{-2(t-10)}\right)u_{10}(t)$$

4. **STEP 1:** Notice the equation is of the form

$$\phi + (t \star \phi) = 1$$

STEP 2: Take Laplace Transforms

$$\begin{aligned}
\mathcal{L}\{\phi\} + \mathcal{L}\{t \star \phi\} &= \mathcal{L}\{1\} \\
\mathcal{L}\{\phi\} + \mathcal{L}\{t\}\mathcal{L}\{\phi\} &= \frac{1}{s} \\
\mathcal{L}\{\phi\} + \frac{1}{s^2}\mathcal{L}\{\phi\} &= \frac{1}{s} \\
\left(1 + \frac{1}{s^2}\right)\mathcal{L}\{\phi\} &= \frac{1}{s} \\
\frac{s^2+1}{s^2}\mathcal{L}\{\phi\} &= \frac{1}{s} \\
\mathcal{L}\{\phi\} &= \left(\frac{s^2}{s^2+1}\right)\left(\frac{1}{s}\right) \\
\mathcal{L}\{\phi\} &= \frac{s}{s^2+1} = \mathcal{L}\{\cos(t)\}
\end{aligned}$$

Hence $\phi(t) = \cos(t)$

5. **STEP 1:**

$$y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

STEP 2: Plug into the ODE

$$y' = a_1 + a_2(2x) + a_3(3x^2) + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = a_2(2) + a_3(3)(2)x + a_4(4)(3)x^2 + \cdots = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

$$xy'' - y' + 4xy = 0$$

$$x\left(\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}\right) - \left(\sum_{n=1}^{\infty} a_n n x^{n-1}\right) + 4x\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-1} - \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^{n+1} = 0$$

$$\sum_{n=1}^{\infty} a_{n+1}(n+1)n x^n - \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=1}^{\infty} 4a_{n-1}x^n = 0$$

We used $m = n-1$, $m = n-1$, and $m = n+1$ respectively

$$\sum_{n=1}^{\infty} a_{n+1}(n+1)n x^n - a_1(1)x^0 - \sum_{n=1}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=1}^{\infty} 4a_{n-1}x^n = 0$$

$$-a_1 + \sum_{n=1}^{\infty}\left[a_{n+1}(n+1)n - a_{n+1}(n+1) + 4a_{n-1}\right]x^n = 0$$

This tells us that $-a_1 = 0 \Rightarrow a_1 = 0$ and

$$\begin{aligned} a_{n+1}(n+1)n - a_{n+1}(n+1) + 4a_{n-1} =& 0 \\ n^2 a_{n+1} + \cancel{na_{n+1}} - \cancel{na_{n+1}} - a_{n+1} + 4a_{n-1} =& 0 \\ \left(n^2 - 1\right) a_{n+1} + 4a_{n-1} =& 0 \\ a_{n+1} =& -\left(\frac{4}{n^2-1}\right) a_{n-1} \end{aligned}$$

Answer: $a_1 = 0$ and $a_{n+1} = -\left(\frac{4}{n^2-1}\right) a_{n-1}$