1. *(10 points)* Let S be a nonempty and bounded subset of $\mathbb{R}$.

Show that there is a sequence (s_n) in S that converges to $\sup(S)$.

2. *(10 points)* Suppose (s_n), (t_n), and (u_n) are sequences in $\mathbb{R}$ with $s_n \to s$, $t_n \to t$, and $u_n \to u$. Show using the **definition** of a limit that

$$s_n + t_n + u_n \to s + t + u$$

Note: Make sure your final answer ends with ϵ

3. *(10 = 7 + 3 points)* Let (s_n) and (t_n) be bounded sequences in $\mathbb{R}$.

(a) Show using the **definition** of $\liminf$ (and without using $\limsup$) that

$$\liminf_{n\to\infty} (s_n + t_n) \geq \left(\liminf_{n\to\infty} s_n\right) + \left(\liminf_{n\to\infty} t_n\right)$$

Hint: First show

$$\inf\{s_n + t_n \mid n > N\} \geq \inf\{s_n \mid n > N\} + \inf\{t_n \mid n > N\}$$

(b) Give an example of bounded sequences (s_n) and (t_n) with

$$\liminf_{n\to\infty} (s_n + t_n) \neq \left(\liminf_{n\to\infty} s_n\right) + \left(\liminf_{n\to\infty} t_n\right)$$

Briefly justify your answer

4. *(10 points)* Show that $(\mathbb{R}^2, d_\infty)$ is complete, where

$$d_\infty((x_1, x_2), (y_1, y_2)) = \max\left\{|x_1 - y_1|, |x_2 - y_2|\right\}$$

Note: You're allowed to use that $\mathbb{R}$ is complete (with its usual absolute value)

5. *(10 = 2 + 8 points)*

(a) (this is quick) Use a convergence test to show that the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

(b) Suppose (s_n) is a sequence such that, for all $n \geq 1$, we have

$$|s_{n+1} - s_n| \leq \frac{1}{n^2}$$

Show that (s_n) converges

Hint: Show that (s_n) is Cauchy. For this, use the Cauchy criterion applied to the series in (a)

6. *(10 points)* Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose there is a sequence (s_n) in $[a, b]$ such that $0 \leq f(s_n) \leq \frac{1}{n}$ for all n Show that there is some $x \in [a, b]$ with $f(x) = 0$

Careful: We don't know whether (s_n) converges!

7. *(10 points)* Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function with the following property:

> There are $C > 0$ and $\alpha > 0$ such that for all $x, y \in \mathbb{R}$, we have
> $$|f(x) - f(y)| \leq C\,|x - y|^{\alpha}$$

Show that f is uniformly continuous on $\mathbb{R}$

8. *(10 points)* Let (S, d) be a compact metric space and suppose $f : S \to \mathbb{R}$ satisfies the following property:

> For all $x \in S$, there are $M > 0$ and $r > 0$ (depending on x) such that, for all $y \in B(x, r)$, $|f(y)| \leq M$

Show directly, using the definition of compactness, that there is $M > 0$ (not depending on x) such that for all $x \in S$, $|f(x)| \leq M$

Careful: Do **NOT** assume that f is continuous.