# MATH 409 – FINAL EXAM

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| Name | |
| Student ID | |
| Signature | |

**Instructions:** This is it, your final hurdle to freedom! You have 120 minutes to take this exam, for a total of 70 points. No books, notes, calculators, or cellphones are allowed. **Please write in complete sentences if you can.** Remember that you are not only graded on your final answer, but also on your work. If you need to continue your work on the back of the page, please clearly indicate so, or else your work may be discarded.

**Academic Honesty Statement:** With the signature above, I certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating (no matter how small) results in an automatic F in the course, and will be further subject to disciplinary consequences, pursuant to the Aggie Honor Code.

*Date*: Wednesday, December 15, 2021.

1. *(10 points)* Prove **one** of the following statements (**NOT** both)

☐ FTC 2: If $f$ is differentiable on $[a, b]$, then

$$\int_a^b f'(x)dx = f(b) - f(a)$$

☐ If $f$ is continuous on $[a, b]$, then $f$ is integrable on $[a, b]$

2. *(10 = 2 + 6 + 2 points)*

(a) Define: $\lim_{n\to\infty} s_n = s$

(b) Suppose $(s_n)$ is a bounded sequence and $(t_n)$ a sequence such that $\lim_{n\to\infty} t_n = \infty$, show that $\lim_{n\to\infty} \frac{s_n}{t_n} = 0$

(c) Give an example where (b) is false if $(s_n)$ is not bounded

3. *(10 = 2 + 4 + 4 points)* Let $(s_n)$ be a bounded sequence of real numbers.

(a) Define: $\liminf_{n\to\infty} s_n$

(b) Use the **definition** of lim inf to show that

$$\liminf_{n\to\infty} s_n = -\left(\limsup_{n\to\infty} -s_n\right)$$

You may use any facts about inf and sup learned in lecture

(c) Prove the result in (b), but this time with subsequences.

**Hint:** First consider a subsequence going to $\liminf_{n\to\infty} s_n$ Then consider a subsequence going to $\limsup_{n\to\infty} -s_n$.

4. *(10 = 2 + 2 + 6 points)*

(a) Define: $(s_n)$ is Cauchy

(b) State the Cauchy criterion for convergence of a series $\sum_{k=1}^{\infty} a_k$

(c) Suppose $(s_n)$ is a sequence such that for all $n$

$$|s_n - s_{n-1}| \leq \left(\frac{1}{2}\right)^n$$

Show that $(s_n)$ converges

**Hint:** Show that $(s_n)$ is Cauchy. For this, use the Cauchy criterion applied to a certain series.

5. *(10 = 2 + 8 points)*

(a) Define: $\lim_{x\to a} f(x) = L$

(b) Use the $\epsilon - \delta$ definition of a limit to show that

$$\lim_{x\to 2} 2x^3 + 3 = 19$$

It might be useful to use $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

6. *(10 = 5 + 5 points)* $f : \mathbb{R} \to \mathbb{R}$ be a function with the following property:

There is $C > 0$ such that for every $x$ and $y$, we have

$$|f(x) - f(y)| \leq C\,|x - y|^2$$

(a) Show $f$ is uniformly continuous on $\mathbb{R}$

(b) Show that if $f$ is differentiable on $\mathbb{R}$, then $f$ must be constant

7. *(10 = 2 + 8 points)*

(a) State the Cauchy criterion for integrability of a function $f$ on $[a, b]$

(b) Use the Cauchy criterion to show that $f(x) = x^2$ is integrable on $[0, 1]$

**Hint:** Let $P$ be the evenly spaced calculus partition of width $\frac{1}{n}$ where $n$ is to be chosen later. At some point, you may need to use $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$