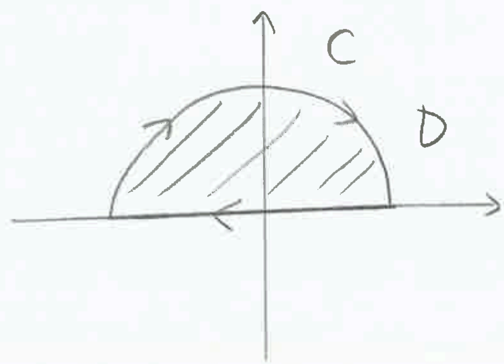
SOLUTIONS

1. *(10 points)* Calculate $\int_C F \cdot dr$ where $F = \frac{4}{3}\langle -y^3, x^3 \rangle$ and C is the upper half circle $x^2+y^2 \leq 9$ with $y \geq 0$, oriented **clockwise**. Simplify your answer.

1) PICTURE (OPTIONAL)



2) BY GREEN'S THEOREM

$$\int_C F \cdot dr = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\,dy$$

$$= \iint_D \left(\frac{4}{3}x^3\right)_x - \left(-\frac{4}{3}y^3\right)_y \, dx\,dy$$

$$= \iint_D 4x^2 + 4y^2 \, dx\,dy$$

$$= \int_0^{\pi}\int_0^3 (4r^2)\, r\, dr\, d\theta = \pi \int_0^3 4r^3\, dr$$

$$= \pi \left[r^4\right]_0^3 = \pi(3^4) = 81\pi$$

Answer	-81π

3) SINCE C IS CLOCKWISE, THE ANSWER IS -81π

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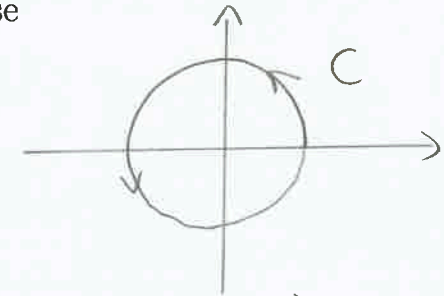
2. *(10 points)* Use parametrizations to find $\int_C F \cdot dr$, where $F = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ and C is the circle centered at $(0,0)$ and radius 2, oriented counterclockwise

1) PICTURE (OPTIONAL)



2) PARAMETRIZE C $\quad r(t) = \langle 2\cos(t), 2\sin(t) \rangle \quad 0 \leq t \leq 2\pi$

3) $\int_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt$

$= \int_0^{2\pi} \left\langle \frac{-2\sin(t)}{4\cos^2(t) + 4\sin^2(t)}, \frac{2\cos(t)}{4\cos^2(t)+4\sin^2(t)} \right\rangle \cdot \langle -2\sin(t), 2\cos(t) \rangle\, dt$

$= \int_0^{2\pi} \left\langle -\frac{2\sin(t)}{4}, \frac{2\cos(t)}{4} \right\rangle \cdot \langle -2\sin(t), 2\cos(t) \rangle\, dt$

$= \int_0^{2\pi} \frac{4\sin^2(t)}{4} + \frac{4\cos^2(t)}{4}\, dt$

$= \int_0^{2\pi} \sin^2(t) + \cos^2(t) = \int_0^{2\pi} 1\, dt = 2\pi$

Answer	2π

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3. *(10 points)* [redacted] find $\int_C F \cdot dr$, where $F = \langle 2xe^{-y}, 2y - x^2e^{-y} \rangle$ and C is the line segment from $(1, 0)$ to $(3, 5)$

1) F CONSERVATIVE ?

$$Q_x - P_y = (2y - x^2e^{-y})_x - (2xe^{-y})_y = \cancel{-2xe^{-y}} + \cancel{2xe^{-y}} = 0$$

2) FIND f

$$F = \nabla f \Rightarrow \langle 2xe^{-y}, 2y - x^2e^{-y} \rangle = \langle f_x, f_y \rangle$$

$$f_x = 2xe^{-y} \Longrightarrow f = \int 2xe^{-y}\,dx = x^2e^{-y} + \text{JUNK}$$

$$f_y = 2y - x^2e^{-y} \Longrightarrow f = \int 2y - x^2e^{-y}\,dy = y^2 + x^2e^{-y} + \text{JUNK}$$

$$f(x,y) = x^2e^{-y} + y^2$$

3) BY FTC,

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(3,5) - f(1,0)$$

$$= (3)^2e^{-5} + 5^2 - (1)^2e^{-0} - 0^2$$

$$= 9e^{-5} + 25 - 1$$

$$= 9e^{-5} + 24$$

Answer	$9e^{-5} + 24$

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$\iint_S 3\sqrt{x+2y+2z}\, dS$

15

4. ([redacted] *points* [redacted]) Find [redacted], where [redacted] and S be the following surface (oriented upwards), parametrized by:

$$\begin{cases} r(u,v) = \left\langle u^2, uv, \frac{1}{2}v^2 \right\rangle \\ \quad 0 \le u \le 1 \\ \quad 0 \le v \le 2 \end{cases}$$

1) SLOPES $\quad r_u = \langle 2u, v, 0 \rangle, \quad r_v = \langle 0, u, v \rangle$

2) $\hat{N} = r_u \times r_v = \begin{vmatrix} i & j & k \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = \langle v^2, -2uv, \underbrace{2u^2}_{\geq 0} \rangle$ ✓

3) $dS = \| r_u \times r_v \| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = \sqrt{(v^2+2u^2)^2}$

$= v^2 + 2u^2$

4) FUNCTION $\quad \sqrt{x+2y+2z}$

$= \sqrt{u^2+2uv+v^2}$

$= \sqrt{(u+v)^2}$

$= 3(u+v)$

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Answer	23

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5. *(15 = 10 + 5 points)*

(a) Find a such that $F = \operatorname{curl}(G)$, where $F = \langle 0, 0, -3x\rangle$ and $G = \langle xy, ax^2, z\rangle$

(b) Use (a) to calculate $\iint_S F \cdot d\mathbf{S}$ where $F = \langle 0, 0, -3x\rangle$ and S is the paraboloid $z = 25 - x^2 - y^2$ above the plane $z = 9$, oriented upwards

(a) $\operatorname{curl}(G) = \begin{vmatrix} i & j & K \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & ax^2 & z \end{vmatrix}$

$= \left\langle \frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(ax^2),\ -\frac{\partial}{\partial x}(z) + \frac{\partial}{\partial z}(xy),\ \frac{\partial}{\partial x}(ax^2) - \frac{\partial}{\partial y}(xy) \right\rangle$

$= \langle 0, 0, \underline{2ax - x}\rangle = \langle 0, 0, \underline{-3x}\rangle$

$2ax - x = -3x \Rightarrow 2a - 1 = -3 \Rightarrow 2a = -2 \Rightarrow \underline{a = -1}$

(TURN PAGE FOR (b))

Answer for (a)	$a = -1$
Answer for (b)	0

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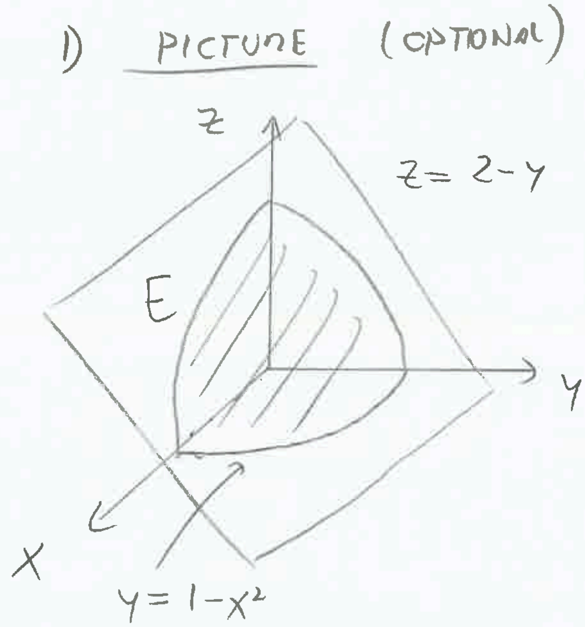
15 $\qquad\qquad\qquad\qquad\qquad\qquad$ $3x - y$

6. (~~points~~ 15 *points*) Find $\iint_S F \cdot d\mathbf{S}$, where $F = \left\langle 3x, -y + e^{x(z^2)}, \sin(xy) \right\rangle$ and S is the boundary of the region E in the first octant bounded by $y = 1 - x^2$ and $y + z = 2$, oriented outwards.

1) PICTURE (OPTIONAL)



2) BY THE DIV THEOREM,

$$\iint_S F \cdot d\vec{S} = \iiint_E \mathrm{DIV}(F)\, dx\,dy\,dz$$

$$= \iiint_E (3x)_x + (-y + e^{x(z^2)})_y + (\sin(xy))_z \, dx\,dy\,dz$$

$$= \iiint_E 3 - 1 + 0 \, dx\,dy\,dz = \iiint_E 2\, dx\,dy\,dz \text{ (SEE BAC}$$

Answer	32/15

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7. *(15 = 5 + 10 points)* Let $F = \langle x, \sin(\sin(y)), \cos(\cos(z)) \rangle$

(a) Show F is conservative

(b) Calculate $\int_C F \cdot dr$, where C is the curve parametrized by $r(t) = \langle t, \sin(t), \sin(t) \rangle$, $0 \leq t \leq \pi$.

Hint: Since $\int_C F \cdot dr$ is independent of path, can you think of a simpler path connecting the start and endpoints?

(a) $$\text{CURL}(F) = \begin{vmatrix} i & j & K \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ X & SN(SN(Y)) & COS(COS(Z)) \end{vmatrix}$$

$$= \Big\langle \frac{\partial}{\partial Y}(COS(COS(Z))) - \frac{\partial}{\partial Z}(SN(SN(Y))),$$

$$-\frac{\partial}{\partial X}(COS(COS(Z))) + \frac{\partial}{\partial Z}(X),$$

$$\frac{\partial}{\partial X}(SN(SN(Y))) - \frac{\partial}{\partial Y}(X) \Big\rangle$$

$= \langle 0-0, -0+0, 0-0 \rangle$

$= \langle 0,0,0 \rangle$ HENCE F IS CONSERVATIVE

Answer for (a)	(show work)
Answer for (b)	$\pi^2/2$

(TURN PAGE FOR (b))

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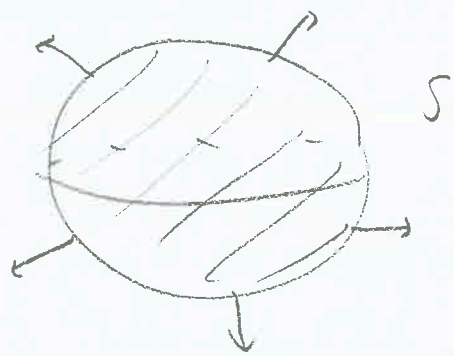
8. *(10 = 5 + 5 points)* The two parts of this problem are independent of each other

   (a) Calculate $\iint_S \operatorname{curl}(F) \cdot d\mathbf{S}$ where $F = \langle x^3, y^4, z^5 \rangle$ and S is the sphere centered at $(0,0,0)$ and radius 3

   (b) Is there a vector field G such that $F = \operatorname{curl} G$ where $F = \langle z, y, x \rangle$? Why or why not?

(a)



SINCE S IS CLOSED BY THE DIVERGENCE THEOREM,

$$\iint_S \operatorname{CURL}(F) \cdot d\vec{S} = \iiint_E \underbrace{\operatorname{DIV}(\operatorname{CURL}(F))}_{0}\, dx\,dy\,dz = \iiint_E 0 = 0$$

(b) No! SUPPOSE $F = \operatorname{CURL}(G)$ THEN $\operatorname{DIV}(F) = \operatorname{DIV}(\operatorname{CURL}(G)) = 0$

BUT $\operatorname{DIV}(F) = (z)_x + (y)_y + (x)_z = 0 + 1 + 0 = 1 \neq 0$

Answer for (a)	0
Answer for (b)	~~(show work)~~ No

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