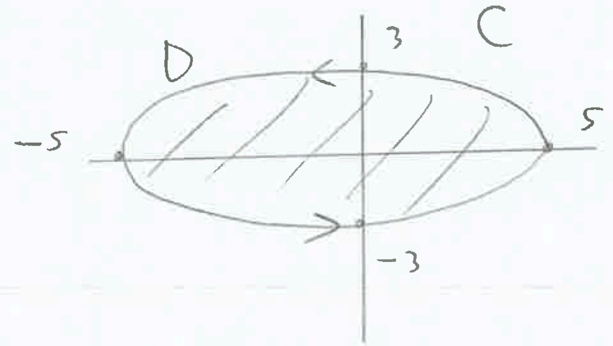
SOLUTIONS

1. *(10 points)* Find the area of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$

1) PICTURE (OPTIONAL)



2) BY THE FORMULA IN THE GREEN'S THEOREM SECTION,

$$\text{AREA}(D) = \frac{1}{2} \int_C x\,dy - y\,dx$$

$$r(t) = \langle 5\cos(t), 3\sin(t) \rangle \quad (0 \leq t \leq 2\pi)$$

$$= \frac{1}{2} \int_0^{2\pi} 5\cos(t)\left(3\cos(t)\right) - 3\sin(t)\left(-5\sin(t)\right) dt$$

$$= \frac{1}{2} \int_0^{2\pi} \underbrace{15\cos^2(t) + 15\cos^2(t)}_{15} dt$$

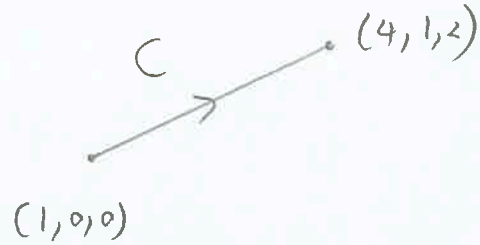
$$= \cancel{\tfrac{1}{2}} \times 15 \times \cancel{2}\pi = 15\pi$$

Answer	15π

☐ Work on back

2. *(10 points)* Calculate $\int_C F \cdot dr$, where $F = \langle z^2, x^2, y^2 \rangle$ and C is the line segment from $(1, 0, 0)$ to $(4, 1, 2)$

1)



2) <u>PARAMETRIZE C</u>

$$x(t) = (1-t)\,1 + t\,(4) = 3t+1$$
$$y(t) = (1-t)\,0 + t\,(1) = t$$
$$z(t) = (1-t)\,0 + t\,(2) = 2t$$

$$r(t) = \langle 3t+1, t, 2t \rangle \qquad 0 \leq t \leq 1$$

3) $$\int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^1 \langle (2t)^2, (3t+1)^2, t^2 \rangle \cdot \langle 3, 1, 2 \rangle \, dt$$

$$= \int_0^1 (4t^2)(3) + (3t+1)^2 + 2t^2\, dt = \int_0^1 12t^2 + 9t^2 + 6t + 1 + 2t^2\, d$$

Answer	$35/3$

$$= \int_0^1 23t^2 + 6t + 1\, dt$$

☐ Work on Scratch Paper

$$= \left[\frac{23}{3} t^3 + 3t^2 + t \right]_0^1 = \frac{23}{3} + 3 + 1 = \frac{23}{3} + \frac{12}{3} = 35/3$$

3. *(10 points)* Find $\int_C F \cdot dr$, where $F = \langle y, x + z\cos(y), \sin(y)\rangle$ and C is the curve parametrized by $r(t) = \langle t, \cos(t), \sin(t)\rangle$, $0 \le t \le \pi$

1) F CONS

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x+z\cos(y) & \sin(y) \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(\sin(y)) - \frac{\partial}{\partial z}(x+z\cos(y)), -\frac{\partial}{\partial x}(\sin(y)) + \frac{\partial}{\partial z}(y), \frac{\partial}{\partial x}(x+z\cos(y)) - \frac{\partial}{\partial y}(y) \right\rangle$$

$$= \langle \cos(y) - \cos(y), 0+0, 1-1 \rangle = \underline{\langle 0,0,0 \rangle} \checkmark$$

2) FIND f

$F = \nabla f \Rightarrow \langle y, x+z\cos(y), \sin(y)\rangle = \langle f_x, f_y, f_z \rangle$

$f_x = y \Rightarrow f = \int y\,dx = xy + \text{JUNK}$

$f_y = x + z\cos(y) \Rightarrow f = \int x + z\cos(y)\,dy = xy + z\sin(y) + \text{JUNK}$

Answer $-\pi$

$f_z = \sin(y)$

$\Rightarrow f = \int \sin(y)\,dz = z\sin(y) + \text{JUNK}$

☐ Work on Scratch Paper

$\underline{f(x,y,z) = xy + z\sin(y)}$ (TURN PAGE)

4. *(15 points)* Find $\iint_S F \cdot d\mathbf{S}$, where $F = \langle -x, +y, z^3 \rangle$ and S is the surface (oriented upwards), parametrized by:

$$\begin{cases} r(u,v) = \langle u\cos(v), u\sin(v), u\rangle \\ 1 \leq u \leq 2 \\ 0 \leq v \leq 2\pi \end{cases}$$

1) $r_U = \langle \cos(V), \sin(V), 1 \rangle$

$r_V = \langle -U\sin(V), U\cos(V), 0 \rangle$

2) $\hat{N} = r_U \times r_V = \begin{vmatrix} i & j & k \\ \cos(V) & \sin(V) & 1 \\ -U\sin(V) & U\cos(V) & 0 \end{vmatrix}$

$= \langle -U\cos(V), -U\sin(V), U\cos^2(V) + U\sin^2(V) \rangle$

$= \langle -U\cos(V), -U\sin(V), \underbrace{U}_{>0} \rangle$ ✓

3) $\iint_S F \cdot d\vec{S} = \iint_D \underbrace{\langle -U\cos(V), U\sin(V), U^3 \rangle}_{F} \cdot \underbrace{\langle -U\cos(V), -U\sin(V), U \rangle}_{\hat{N}} dA$

$= \int_0^{2\pi}\int_1^2 U^2\cos^2(V) - U^2\sin^2(V) + U^4 \, dU\, dV$

$= \int_0^{2\pi}\int_1^2 U^2\cos(2V) + U^4 \, dU\, dV$

Answer	$62\pi/5$

$= \int_0^{2\pi} \left[\frac{1}{3}U^3\cos(2V) + \frac{U^5}{5} \right]_1^2 dV$

☐ Work on back

$= \int_0^{2\pi} \frac{8}{3}\cos(2V) - \frac{1}{3}\cos(2V) + \left(\frac{32}{5}\right) - \left(\frac{1}{5}\right) dV = \int_0^{2\pi} \cancel{\frac{7}{3}\cos(2V)} + \frac{31}{5}\, dV$

$= \left(\frac{31}{5}\right)(2\pi) = \frac{62\pi}{5}$

5. *(15 points)* Calculate $\int_C F \cdot dr$, where

$$F = \langle x + y^2, y + z^2, z + x^2 \rangle$$

And C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, oriented counterclockwise from above.

Hint: C lies on a plane

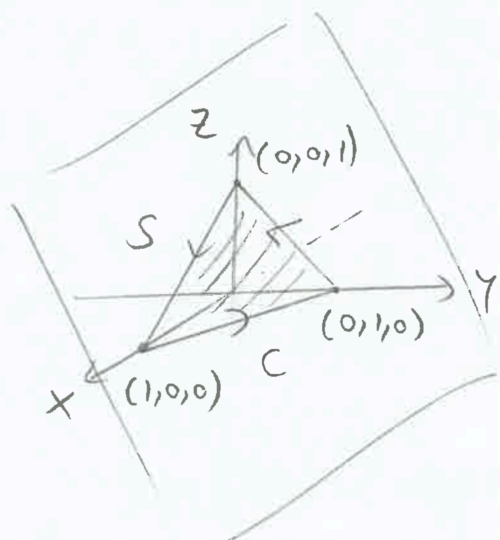
1) $\operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix}$

$= \langle \frac{\partial}{\partial y}(z+x^2) - \frac{\partial}{\partial z}(y+z^2), -\frac{\partial}{\partial x}(z+x^2) + \frac{\partial}{\partial z}(x+y^2),$
$\frac{\partial}{\partial x}(y+z^2) - \frac{\partial}{\partial y}(x+y^2) \rangle$

$= \langle -2z, -2x, -2y \rangle$

2) BY STOKES, $\int_C F \cdot dr = \iint_S \operatorname{curl}(F) \cdot d\vec{S}$

3) WHAT IS S ?



Answer -1

LET S = INSIDE OF C

☐ Work on back

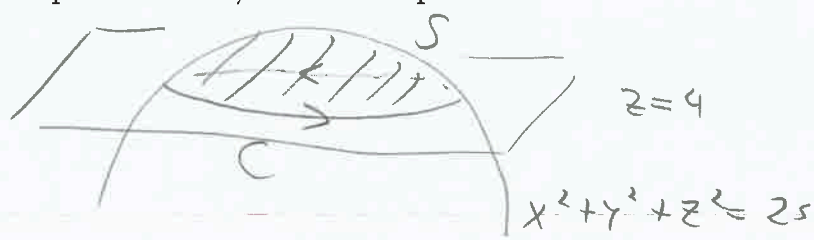
NOTICE S LIES INSIDE THE PLANE $x+y+z=1 \Rightarrow z = 1-x-y$

6. *(15 points)* Calculate $\iint_S \operatorname{curl}(F) \cdot d\mathbf{S}$, where

$$F = \langle z^3 + y^3, x^3 + z^3, y^3 + x^3 \rangle$$

And S is the part of the sphere $x^2 + y^2 + z^2 = 25$ strictly above the plane $z = 4$, oriented upwards

1)



BY STOKES, $\iint_S \operatorname{curl}(F) \cdot d\vec{S} = \int_C F \cdot dr$

NOTICE $x^2+y^2+z^2 = 25$ AND $z=4$ GIVES $x^2+y^2+16 = 25$

$\Rightarrow$ $x^2+y^2=9$

SO C IS A CIRCLE OF RADIUS 3 IN THE PLANE $z=4$

$r(t) = \langle 3\cos(t), 3\sin(t), 4 \rangle$ $\quad 0 \le t \le 2\pi$

2) $\int_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt$

$= \int_0^{2\pi} \langle 4^3 + (3\sin(t))^3, (3\cos(t))^3 + 4^3, (3\sin(t))^3 + (3\cos(t))^3 \rangle \cdot \langle -3\sin(t), 3\cos(t), 0 \rangle\, dt$

Answer | 0

$= \int_0^{2\pi} -3(4^3)\cancel{\sin(t)}^{0} - 3^4 \sin^4(t) + 3^4\cos^4(t) + (3)(4^3)\cancel{\cos(t)}^{0}\, dt$

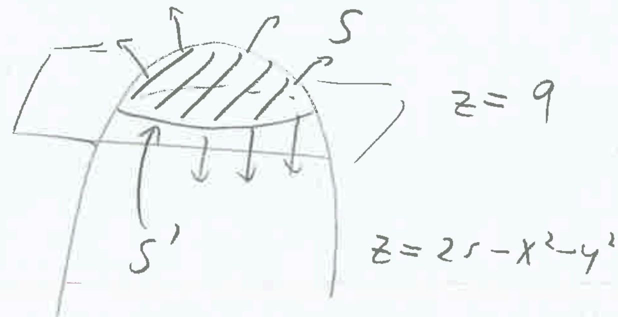
☐ Work on back

(TURN PAGE)

7. (20 *points)* Find $\iint_S F \cdot d\vec{S}$ where $F = \langle x+z, x+2y, x^2 \rangle$ and S is part of the surface $z = 25 - x^2 - y^2$ strictly above the plane $z = 9$

Warning: S is not closed

1)



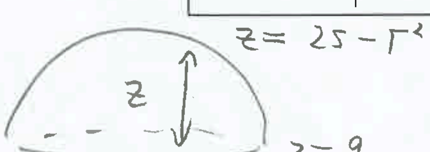
LET S' = BOTTOM DISK OF S, THEN $S + S'$ IS CLOSED, SO BY THE DIVERGENCE THEOREM

$$\iint_{S+S'} F \cdot d\vec{S} = \iiint_E \mathrm{DIV}(F)\, dx\,dy\,dz$$

$$= \iiint_E (x+z)_x + (x+y)_y + (x^2+z)_z \, dx\,dy\,dz$$

$$= \iiint_E 1 + 1 + 1 \, dx\,dy\,dz = \iiint_E 3 \, dx\,dy\,dz$$

Answer	448π



$$\begin{cases} 9 \leq z \leq 25 - r^2 \\ 0 \leq r \leq 4 \longleftarrow \\ 0 \leq \theta \leq 2\pi \end{cases}$$

☐ Work on back

$z = 25 - r^2$ AND $z = 9$ GIVES $25 - r^2 = 9$ $\Rightarrow r^2 = 16 \Rightarrow r = 4$

8. *(10 points)* Find the surface area of the surface S that has a parametrization of the form $r(u,v)$ with $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$ whose normal vector is

$$\hat{N} = a(b + a\cos(v))\,\langle \cos(v)\cos(u), \cos(v)\sin(u), \sin(v)\rangle$$

Where $0 < a < b$ are fixed constants.

You don't need to find $r(u,v)$ to solve this problem.

1) $dS = \|\hat{N}\| = \|a(\underbrace{b + a\cos(V)}_{\geq b + a(-1) = b-a > 0}) \langle \cos(V)\cos(U), \cos(V)\sin(U), \sin(V)\rangle$

$= a(b + a\cos(V)) \, \| \langle \cos(V)\cos(U), \cos(V)\sin(U), \sin(V)\rangle \|$

$= a(b + a\cos(V)) \left(\cos^2(V)\cos^2(U) + \cos^2(V)\sin^2(U) + \sin^2(V) \right)^{\frac{1}{2}}$

$= a(b + a\cos(V)) \left(\cos^2(V) \underbrace{[\cos^2(U) + \sin^2(U)]}_{1} + \sin^2(V) \right)^{\frac{1}{2}}$

$= a(b + a\cos(V)) \left(\underbrace{\cos^2(V) + \sin^2(V)}_{1} \right)^{\frac{1}{2}}$

$= a(b + a\cos(V))$

$= 2\pi a \left[bV + a\cancel{\sin(V)}^{\,0} \right]_0^{2\pi}$

$= (2\pi a)(2\pi b) = 4\pi^2 ab$

Answer $4\pi^2 ab$

2) $\text{AREA}(S) = \iint_D dS$

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$= \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos(V))\,dU\,dV = 2\pi \int_0^{2\pi} a(b + a\cos(V))\,dV$