1.

Let $M = \sup(S)$. Then, for every n, $M - \frac{1}{n} < M = \sup(S)$, and therefore, by definition of sup, there is $s_n \in S$ such that $s_n > M - \frac{1}{n}$. But since M is an upper bound for S, we also have $s_n \leq M$, and therefore $M - \frac{1}{n} < s_n \leq M$

Since $M - \frac{1}{n} \to M$ and $M \to M$, by the squeeze theorem, we have $s_n \to M = \sup(S)$ $\square$

2.

Let $\epsilon > 0$ be given.

Since $s_n \to s$, there is N_1 such that if $n > N_1$, then $|s_n - s| < \frac{\epsilon}{3}$

Since $t_n \to t$, there is N_2 such that if $n > N_2$, then $|t_n - t| < \frac{\epsilon}{3}$

Since $u_n \to u$, there is N_3 such that if $n > N_3$, then $|u_n - u| < \frac{\epsilon}{3}$

Let $N = \max\{N_1, N_2, N_3\}$, then if $n > N$, we get:

$$\begin{aligned}
|s_n + t_n + u_n - (s + t + u)| &= |s_n - s + t_n - t + u_n - u| \\
&\leq |s_n - s| + |t_n - t| + |u_n - u| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon \checkmark
\end{aligned}$$

Hence $s_n + t_n + u_n \to s + t + u$ $\square$

3.

(a) Let N be given, then if $n > N$, we have

$$s_n \geq \inf \{s_n \mid n > N\}$$

And

$$t_n \geq \inf \{t_n \mid n > N\}$$

And therefore

$$s_n + t_n \geq \inf \{s_n \mid n > N\} + \inf \{t_n \mid n > N\}$$

Since $n > N$ was arbitrary, taking the inf over all $n > N$ on the left hand side, we get

$$\inf \{s_n + t_n \mid n > N\} \geq \inf \{s_n \mid n > N\} + \inf \{t_n \mid n > N\}$$

And therefore

$$\begin{aligned}
\liminf_{n\to\infty} (s_n + t_n) &= \lim_{N\to\infty} \inf \{s_n + t_n \mid n > N\} \\
&\overset{(a)}{\geq} \lim_{N\to\infty} \inf \{s_n \mid n > N\} + \inf \{t_n \mid n > N\} \\
&= \lim_{N\to\infty} \inf \{s_n \mid n > N\} + \lim_{N\to\infty} \inf \{t_n \mid n > N\} \\
&= \left(\liminf_{n\to\infty} s_n\right) + \left(\liminf_{n\to\infty} t_n\right)
\end{aligned}$$

(b) Let $s_n = (-1)^n$ and $t_n = -s_n = (-1)^{n+1}$

Then

$$\left(\liminf_{n\to\infty} s_n\right) + \left(\liminf_{n\to\infty} t_n\right) = -1 + (-1) = -2$$

But

$$\liminf_{n\to\infty} s_n + t_n = \liminf_{n\to\infty} 0 = 0 \neq -2$$

4. Let $(x^{(n)}) = (x_1^{(n)}, x_2^{(n)})$ be a Cauchy sequence in $\mathbb{R}^2$

Claim: $x_1^{(n)}$ and $x_2^{(n)}$ are Cauchy in $\mathbb{R}$

Proof: Let $\epsilon > 0$ be given, then there is N such that if $m, n > N$, then $d_\infty(x^{(m)}, x^{(n)}) < \epsilon$, that is

$$\max\left\{\left|x_1^{(m)} - x_1^{(n)}\right|, \left|x_2^{(m)} - x_2^{(n)}\right|\right\} < \epsilon$$

With that same N, if $m, n > N$, then

$$\left|x_1^{(m)} - x_1^{(n)}\right| \leq \max\left\{\left|x_1^{(m)} - x_1^{(n)}\right|, \left|x_2^{(m)} - x_2^{(n)}\right|\right\} < \epsilon \checkmark$$

And similarly $\left|x_2^{(m)}, x_2^{(n)}\right| < \epsilon$. $\square$

Since $x_1^{(n)}$ is Cauchy and $\mathbb{R}$ is complete, $x_1^{(n)} \to x_1$ for some $x_1 \in \mathbb{R}$, and similarly $x_2^{(n)} \to x_2$ for some $x_2 \in \mathbb{R}$. Let $x =: (x_1, x_2)$

Claim: $x^{(n)} \to x$

Let $\epsilon > 0$ be given. Then, since $x_1^{(n)} \to x_1$, there is N_1 such that if $n > N_1$, then $\left|x_1^{(n)} - x_1\right| < \epsilon$, and similarly there is N_2 such that if $n > N_2$, then $\left|x_2^{(n)} - x_2\right| < \epsilon$.

Let $N = \max\{N_1, N_2\}$, then if $n > N$, we have

$$d_\infty(x^{(n)}, x) = \max\left\{\left|x_1^{(n)} - x_1\right|, \left|x_2^{(n)} - x_2\right|\right\} < \max\{\epsilon, \epsilon\} = \epsilon \checkmark$$

And therefore $x^{(n)} \to x$ $\square$

5. (a) Let $f(x) = \frac{1}{x^2}$, then f is ≥ 0 and decreasing and

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^\infty = 0 + 1 = 1 < \infty$$

Therefore by the Integral Test, $\sum_{n=1}^\infty \frac{1}{n^2}$ converges

(b) Let $\epsilon > 0$ be given. Then since $\sum_{n=1}^\infty \frac{1}{n^2}$ converges, it satisfies the Cauchy criterion, and hence there is N such that if $n \geq m > N$, then

$$\left| \sum_{k=m}^{n} \frac{1}{k^2} \right| < \epsilon$$

But then, with the same N, if $m, n > N$, WLOG, $n > m$, and therefore

$$\begin{aligned}
|s_m - s_n| &= |s_m - s_{m+1} + s_{m+1} - s_{m+2} + \cdots + s_{n-1} - s_n| \\
&\leq |s_m - s_{m+1}| + |s_{m+1} - s_{m+2}| + \cdots + |s_{n-1} - s_n| \\
&\leq \frac{1}{m^2} + \frac{1}{(m+1)^2} + \cdots + \frac{1}{(n-1)^2} \\
&= \sum_{k=m}^{n-1} \frac{1}{k^2} \\
&< \sum_{k=m}^{n} \frac{1}{k^2} \\
&< \epsilon \checkmark
\end{aligned}$$

Hence (s_n) is Cauchy, and since $\mathbb{R}$ is complete, (s_n) converges $\square$

6.

BOLZANO-WEIERSTRAß TIME!!!

Since (s_n) is a sequence in $[a, b]$, (s_n) is bounded, and therefore, by the Bolzano-Weierstraß Theorem, (s_n) has a convergent subsequence (s_{n_k}) that converges to some $x \in [a, b]$.

Since $s_{n_k} \to x$ and f is continuous, $f(s_{n_k}) \to f(x)$.

But, on the other hand, since $0 \leq f(s_n) \leq \frac{1}{n}$, we have $0 \leq f(s_{n_k}) \leq \frac{1}{n_k}$, and $\frac{1}{n_k} \to 0$, by the squeeze theorem, we have $f(s_{n_k}) \to 0$.

Combining $f(s_{n_k}) \to f(x)$ and $f(s_{n_k}) \to 0$, we get $f(x) = 0$ $\square$

7.

STEP 1: Scratchwork:

$$|f(x) - f(y)| \leq C\,|x-y|^{\alpha} < \epsilon \Rightarrow |x-y|^{\alpha} < \frac{\epsilon}{C} \Rightarrow |x-y| < \left(\frac{\epsilon}{C}\right)^{\frac{1}{\alpha}}$$

STEP 2: Actual Proof:

Let $\epsilon > 0$ be given, let $\delta = \left(\frac{\epsilon}{C}\right)^{\frac{1}{\alpha}}$, then if $x, y \in \mathbb{R}$ with $|x-y| < \delta$, then

$$|f(x) - f(y)| \leq C\,|x-y|^{\alpha} < C\left[\left(\frac{\epsilon}{C}\right)^{\frac{1}{\alpha}}\right]^{\alpha} = C\left(\frac{\epsilon}{C}\right) = \epsilon \checkmark$$

Therefore f is uniformly continuous on $\mathbb{R}$ $\square$

8.

By assumption, for all $x \in S$, there are $M = M(x) > 0$ and $r = r(x) > 0$ such that $|f(y)| \leq M(x)$ for all $y \in B(x, r(x))$.

Now consider the following family of sets

$$\mathcal{U} = \{B(x, r(x)) \mid x \in S\}$$

Each $B(x, r(x))$ is open (by definition) and each $x \in S$ is in $B(x, r(x))$, hence $\mathcal{U}$ is an open cover of S.

But since S is compact, $\mathcal{U}$ has a finite sub-cover

$$\mathcal{V} = \{B(x_1, r(x_1)), \dots, B(x_N, r(x_N))\}$$

Let $M = \max\{M(x_1), \dots M(x_N)\}$ (which is independent of x)

Then for all $x \in S$, since $\mathcal{V}$ covers S, we have $x \in B(x_n, r(x_n))$ for some n, and therefore, by definition

$$|f(x)| \leq M(x_n) \leq M \Rightarrow |f(x)| \leq M \checkmark \quad \square$$