MATH 409 – FINAL EXAM – SOLUTIONS

1. **Option 1:** Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be any partition of $[a, b]$

   Then by the Mean Value Theorem, on each sub-piece $[t_{k-1}, t_k]$, there is x_k such that

$$f'(x_k) = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}} \Rightarrow f'(x_k)(t_k - t_{k-1}) = f(t_k) - f(t_{k-1})$$

   But with that choice of x_k, the Riemann sum $R(f', P)$ becomes:

$$\begin{aligned}\sum_{k=1}^{n} f'(x_k)(t_k - t_{k-1}) =& \sum_{k=1}^{n} f(t_k) - f(t_{k-1}) \\ =& f(t_1) - f(t_0) + f(t_2) - f(t_1) + \cdots + f(t_n) - f(t_{n-1}) \\ =& f(t_n) - f(t_0) \\ =& f(b) - f(a)\end{aligned}$$

   Since this is true for every partition, it follows that $\int_a^b f'(x)dx = f(b) - f(a)$ $\square$

Date: Wednesday, December 15, 2021.

Option 2: Let $\epsilon > 0$ be given

Since f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$. Therefore there is $\delta > 0$ such that for every x, y in $[a, b]$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\epsilon}{b-a}$

For this δ, let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be any partition with $\text{mesh}(P) < \delta$

Then on each sub-piece, since f is continuous, by the Extreme Value Theorem, $M(f, [t_{k-1}, t_k]) = f(x_k)$ and $m(f, [t_{k-1}, t_k]) = f(y_k)$ for some x_k and y_k. And since $|x_k - y_k| < t_k - t_{k-1} < \delta$, we have $f(x_k) - f(y_k) < \frac{\epsilon}{b-a}$ and so

$$
\begin{aligned}
U(f, P) - L(f, P) &= \sum_{k=1}^{n} \left(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])\right)(t_k - t_{k-1}) \\
&= \sum_{k=1}^{n} \left(f(x_k) - f(y_k)\right)(t_k - t_{k-1}) \\
&\leq \sum_{k=1}^{n} \frac{\epsilon}{b-a}(t_k - t_{k-1}) \\
&= \frac{\epsilon}{b-a} \sum_{k=1}^{n} t_k - t_{k-1} \\
&= \frac{\epsilon}{b-a}(t_1 - t_0 + t_2 - t_1 + \cdots + t_n - t_{n-1}) \\
&= \frac{\epsilon}{b-a}(t_n - t_0) = \frac{\epsilon}{b-a}(b - a) = \epsilon
\end{aligned}
$$

Therefore, by the Cauchy criterion, f is integrable on $[a, b]$ $\square$

2. (a) For all $\epsilon > 0$ there is N such that if $n > N$ then $|s_n - s| < \epsilon$

(b) **STEP 1: Scratchwork**

$$\left|\frac{s_n}{t_n} - 0\right| = \frac{|s_n|}{|t_n|} \leq \frac{M}{|t_n|} < \epsilon \Rightarrow |t_n| > \frac{M}{\epsilon}$$

STEP 2: Let $\epsilon > 0$ be given. Since s_n is bounded, $|s_n| \leq M$ for some M. Since $t_n \to \infty$ there is N such that if $n > N$ then $t_n > \frac{M}{\epsilon}$

With that N, if $n > N$, then

$$\left|\frac{s_n}{t_n} - 0\right| = \frac{|s_n|}{|t_n|} \leq \frac{M}{|t_n|} < \frac{M}{\frac{M}{\epsilon}} = \epsilon$$

Therefore $\lim_{n\to\infty} \frac{s_n}{t_n} = 0$ ✓

(c) Let $s_n = n$ and $t_n = n \to \infty$, then

$$\frac{s_n}{t_n} = \frac{n}{n} = 1$$

Which does not converge to 0

3. (a)

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \inf\{s_n \mid n > N\}$$

(b) From lecture, we know that for any set S, we have

$$\inf(S) = -\sup(-S)$$

Now apply the identity above with $S = \{s_n \mid n > N\}$ to get

$$\inf\{s_n \mid n > N\} = -\sup\{-s_n \mid n > N\}$$

Finally let $N \to \infty$ to get

$$\lim_{N\to\infty} \inf\{s_n \mid n > N\} = -\lim_{N\to\infty} \sup\{-s_n \mid n > N\}$$

$$\liminf_{n\to\infty} s_n = -\left(\limsup_{n\to\infty} -s_n\right)$$

(c) **STEP 1:** Let (s_{n_k}) be a subsequence of (s_n) going to $s =: \liminf_{n\to\infty} s_n$. Then $-s_{n_k}$ is a subsequence of $(-s_n)$ converging to $-s$. Since $\limsup_{n\to\infty}$ is the largest limit point of $(-s_n)$ we get

$$\limsup_{n\to\infty} -s_n \geq -s = -\liminf_{n\to\infty} s_n$$

Now let $(-s_{n_k})$ be a subsequence of $(-s_n)$ going to $t =: \limsup_{n\to\infty} -s_n$. Then s_{n_k} is a subsequence of (s_n) going to $-t$, but since $\liminf_{n\to\infty} s_n$ is the smallest possible limit point of (s_n) we get

$$\liminf_{n\to\infty} s_n \leq -t = -\left(\limsup_{n\to\infty} -s_n\right)$$

Combining both terms we get our desired result □

4. (a) For all $\epsilon > 0$ there is N such that for all m, n if $m, n > N$ then $|s_m - s_n| < \epsilon$

(b) For all $\epsilon > 0$ there is N such that if $n \geq m > N$ we have $|\sum_{k=m}^{n} a_k| < \epsilon$

(c) Let $\epsilon > 0$ be given. Then since $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$ converges (it's a geometric series with $\left|\frac{1}{2}\right| < 1$), by the Cauchy criterion, there is N such that if $n \geq m > N$ then $\left|\sum_{k=m}^{n} \left(\frac{1}{2}\right)^k\right| < \epsilon$.

With that N, if $n \geq m > N$, then

$$\begin{aligned}
|s_n - s_m| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \cdots + s_{m+1} - s_m| \\
&\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \cdots + |s_{m+1} - s_m| \\
&\leq \left(\frac{1}{2}\right)^n + \cdots + \left(\frac{1}{2}\right)^{m+1} \\
&\leq \sum_{k=m}^{n} \left(\frac{1}{2}\right)^k \\
&< \epsilon
\end{aligned}$$

Therefore (s_n) is Cauchy and therefore (s_n) converges.

5. (a) For all $\epsilon > 0$ there is $\delta > 0$ such that for all x, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$

(b) **STEP 1: Scratch work**

$$|f(x) - L| = \left|2x^3 + 3 - 19\right| = \left|2x^3 - 16\right| = 2\left|x^3 - 8\right| = 2\left|x - 2\right| x^2 + 2x + 4$$

But if $|x - 2| < 1$ then

$$|x| = |x - 2 + 2| \leq |x - 2| + |2| < 1 + 2 = 3$$

Hence: $\left|x^2 + 2x + 4\right| \leq |x|^2 + 2|x| + 4 \leq 3^2 + 2(3) + 4 = 9 + 6 + 4 = 19$

Therefore $|f(x) - L| \leq 2|x - 2|(19) = 38|x - 2| < \epsilon$

Which gives $|x - 2| < \frac{\epsilon}{38}$

STEP 2: Let $\epsilon > 0$ be given, let $\delta = \min\left\{1, \frac{\epsilon}{38}\right\}$, then if $0 < |x - 2| < \delta$, then $|x - 2| < 1$ and so $|x| < 3$ and $\left|x^2 + 2x + 4\right| < 19$, and also $|x - 2| < \frac{\epsilon}{38}$ and so

$$|f(x) - 19| = 2|x - 2|\left|x^2 + 2x + 4\right| \leq 2|x - 2|(19) = 38|x - 2| < 38\left(\frac{\epsilon}{38}\right) = \epsilon \checkmark$$

And therefore $\lim_{x\to 2} 2x^3 + 3 = 19$

6. (a) **STEP 1: Scratchwork**

$$|f(x) - f(y)| \leq C|x-y|^2 < \epsilon \Rightarrow |x-y|^2 < \frac{\epsilon}{C} \Rightarrow |x-y| < \sqrt{\frac{\epsilon}{C}}$$

Which suggests to let $\delta = \sqrt{\frac{\epsilon}{C}}$

STEP 2: Actual Proof:

Let $\epsilon > 0$ be given, let $\delta = \sqrt{\frac{\epsilon}{C}}$, then if $|x-y| < \delta$, then

$$|f(x) - f(y)| \leq C|x-y|^2 < C\left(\sqrt{\frac{\epsilon}{C}}\right)^2 = C\frac{\epsilon}{C} = \epsilon \checkmark$$

Hence f is uniformly continuous on $\mathbb{R}$

(b) Notice that the above identity with $y = x + h$ implies that

$$|f(x+h) - f(x)| \leq C(x+h-x)^2 = Ch^2$$

$$\text{Therefore } \left|\frac{f(x+h) - f(x)}{h}\right| \leq Ch$$

But since $\lim_{h\to 0} Ch = 0$, by the squeeze theorem, we get $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 0$ that is $f'(x) = 0$ for all x, so f is constant.

7. (a) For all $\epsilon > 0$ there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$

(b) Let $\epsilon > 0$ be given, let let n be TBA, and let P be the evenly spaced (calculus) partition of $[0,1]$ with $t_k = \frac{k}{n}$

Then since x^2 is increasing, we have

$$m(f,[t_{k-1},t_k]) = f(t_{k-1}) = (t_{k-1})^2 = \left(\frac{k-1}{n}\right)^2 = \frac{(k-1)^2}{n^2}$$

$$M(f,[t_{k-1},t_k]) = f(t_k) = (t_k)^2 = \left(\frac{k}{n}\right)^2 = \frac{k^2}{n^2}$$

And therefore

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{k=1} \left(M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\right)(t_k - t_{k-1}) \\
&= \sum_{k=1}^{n} \left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2}\right)\left(\frac{1}{n}\right) \\
&= \sum_{k=1}^{n} \frac{k^2 - (k-1)^2}{n^3} \\
&= \sum_{k=1}^{n} \frac{k^2 - k^2 + 2k - 1}{n^3} \\
&= \sum_{k=1}^{n} \frac{2k-1}{n^3}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n^3}\left(2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1\right) \\
&= \frac{1}{n^3}\left(2\left(\frac{n(n+1)}{2}\right) - n\right) \\
&= \frac{1}{n^3}\left(n^2 + n - n\right) \\
&= \frac{n^2}{n^3} \\
&= \frac{1}{n}
\end{aligned}
$$

But now let n be large enough such that $\frac{1}{n} < \epsilon$ and we get

$$U(f,P) - L(f,P) = \frac{1}{n} < \epsilon$$

Therefore f is integrable on $[0,1]$

Note: Alternatively, in the calculation above, you may notice that the sum above is telescoping, and equal to:

$$U(f,P) - L(f,P) = \frac{1}{n^3}\sum_{k=1}^{n} k^2 - (k-1)^2 = \frac{1}{n^3}\left(n^2 - 0^2\right) = \frac{1}{n}$$