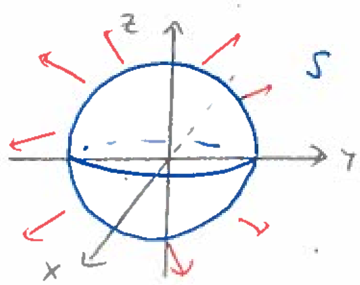
SOLUTIONS

1. *(10 points)* Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the sphere of radius 2 centered at $(0, 0, 0)$, oriented outwards, and $\mathbf{F} = \langle y^2 x, z^2 y, x^2 z \rangle$. Include a picture of S and its orientation.

1) PICTURE



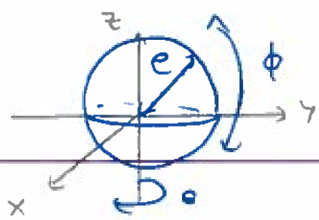
2) DIVERGENCE THEOREM (SINCE S IS CLOSED)

$$\iint_S F \cdot d\vec{S} = \iiint_E \mathrm{DIV}(F)\, dx\,dy\,dz \qquad E = \text{BALL OF RADIUS } 2$$

$$\mathrm{DIV}(F) = (y^2x)_x + (z^2y)_y + (x^2z)_z = y^2 + z^2 + x^2 = x^2+y^2+z^2$$

so $$\iint_S F \cdot d\vec{S} = \iiint_E x^2+y^2+z^2 \, dx\,dy\,dz$$

3)



$0 \leq \rho \leq 2$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

SPHERICAL COORDINATES

$$\iint_S F \cdot d\vec{S} = \int_0^{\pi}\int_0^{2\pi}\int_0^{2} \rho^2 \, \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$

$$= \left(\int_0^2 \rho^4 \, d\rho\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin(\phi)\, d\phi\right)$$

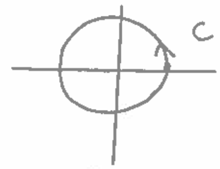
$$= \left[\frac{1}{5}\rho^5\right]_0^2 (2\pi) \left[-\cos(\phi)\right]_0^{\pi}$$

$$= \frac{1}{5}(32)(2\pi)(2)$$

$$= \frac{128\pi}{5}$$

2. *(10 points)* Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle x - y^3, 2x + x^3 \rangle$, where C is the circle centered at $(0,0)$ and radius 2, oriented counterclockwise.

1) PICTURE



(OPTIONAL)

2) F CONSERVATIVE ?

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (2x + x^3)_x - (x - y^3)_y = 2 + 3x^2 - (-3y^2)$$

$$= 2 + 3x^2 + 3y^2 \neq 0$$

3) GREEN'S THEOREM

$$\int_C F \cdot dr = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\,dy$$

$$= \iint_D 2 + 3(x^2 + y^2) \, dx\,dy$$

D = DISK OF RADIUS 2

$$= \int_0^{2\pi} \int_0^2 (2 + 3r^2)\, r \, dr\, d\theta$$

$$= 2\pi \left(\int_0^2 2r + 3r^3 \, dr \right)$$

$$= 2\pi \left[r^2 + \frac{3}{4} r^4 \right]_0^2$$

$$= 2\pi \left(4 + \frac{3}{4}(16) \right)$$

$$= 2\pi (4 + 12) = 2\pi (16)$$

$$= 32\pi$$

(NOTE VERSION B : 40π)

OTHER SOL DO DIRECTLY (BOO!)

$$\int_C F\cdot dr = \int_1^2 \langle t^2 t^3 e^{t^6}, t t^3 e^{t^6}, t t^2 e^{t^6}\rangle \cdot \langle 1, 2t, 3t^2\rangle\, dt$$

$$= \int_1^2 t^5 e^{t^6} + 2t^5 e^{t^6} + 3t^5 e^{t^6} = \int_1^2 6t^5 e^{t^6}$$

$$= \left[e^{t^6}\right]_1^2$$

$$= e^{64} - e$$

3. *(10 points)* Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ with $1 \le t \le 2$, and $\mathbf{F} = \langle yze^{xyz}, xze^{xyz}, xye^{xyz}\rangle$.

1) F CONSERVATIVE?

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yze^{xyz} & xze^{xyz} & xye^{xyz} \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(xye^{xyz}) - \frac{\partial}{\partial z}(xze^{xyz}), -\frac{\partial}{\partial x}(xye^{xyz}) + \frac{\partial}{\partial z}(yze^{xyz}), \frac{\partial}{\partial x}(xze^{xyz}) - \frac{\partial}{\partial y}(yze^{xyz}) \right\rangle$$

$$= \langle \cancel{xe^{xyz}} + \cancel{xyxz e^{xyz}} - \cancel{xe^{xyz}} - \cancel{xz xy e^{xyz}},$$
$$- \cancel{ye^{xyz}} - \cancel{xyyz e^{xyz}} + \cancel{ye^{xyz}} + \cancel{yz xy e^{xyz}},$$
$$\cancel{ze^{xyz}} + \cancel{xzyz e^{xyz}} - \cancel{ze^{xyz}} - \cancel{yz xz e^{xyz}} \rangle$$

$= \langle 0,0,0\rangle$ (GREAT SUCCESS!) $\Rightarrow$ F IS CONS!

2) FIND f $\quad \nabla f = F \Rightarrow \langle f_x, f_y, f_z\rangle = \langle yze^{xyz}, xze^{xyz}, xye^{xyz}\rangle$

$$f_x = yze^{xyz} \Rightarrow f = \int yze^{xyz}\,dx = \cancel{yz}\frac{1}{\cancel{yz}} e^{xyz} = e^{xyz} + \text{JUNK}$$

$$f_y = xze^{xyz} \Rightarrow f = \int xze^{xyz}\,dy = \cancel{xz}\frac{1}{\cancel{xz}} e^{xyz} = e^{xyz} + \text{JUNK}$$

$$f_z = xye^{xyz} \Rightarrow f = \int xye^{xyz}\,dz = \cancel{xy}\frac{1}{\cancel{xy}} e^{xyz} = e^{xyz} + \text{JUNK}$$

$$\Rightarrow f(x,y,z) = e^{xyz}$$

3) FTC FOR LINE INTEGRALS

$$\int_C F\cdot dr = \int_C \nabla f\cdot dr = f(r(2)) - f(r(1))$$

$r(2) = \langle 2, 2^2, 2^3\rangle = \langle 2,4,8\rangle$

$r(1) = \langle 1, 1^2, 1^3\rangle = \langle 1,1,1\rangle$

$$= f(2,4,8) - f(1,1,1)$$

$$= e^{(2)(4)(8)} - e^{(1)(1)(1)} = e^{64} - e$$

(VERSION B: $e^{64} - 1$)

OTHER SOL DO DIRECTLY : $r(t) = \langle 5\cos(t), 5\sin(t), 4 \rangle$

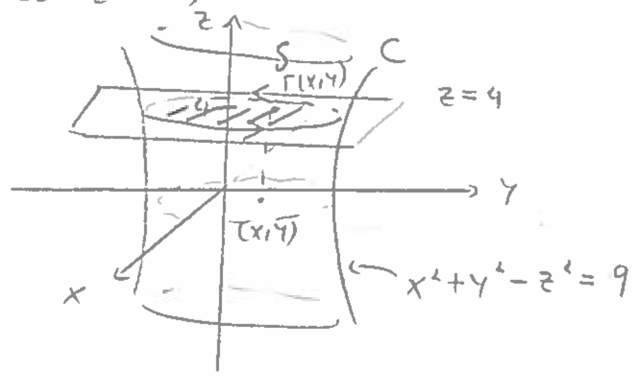
$$\int_C F \cdot dr = \int_0^{2\pi} \langle \underbrace{5\cos(t)}_{x}\,\underbrace{5\sin(t)}_{y},\ \underbrace{5\sin(t)}_{y}\,\underbrace{4}_{z},\ \underbrace{5\cos(t)}_{x}\,\underbrace{4}_{z} \rangle \cdot \langle -5\sin(t), 5\cos(t), 0 \rangle\, dt$$

$$= \int_0^{2\pi} -125\sin^2(t)\cos(t) + 100\sin(t)\cos(t)\, dt = 0$$

4. *(10 points)* Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle xy, yz, xz \rangle$, and C (oriented counterclockwise) is the curve of intersection of the surfaces $\underbrace{x^2 + y^2 - z^2 = 9}$ and $z = 4$. Include a picture of C and the surfaces.

HYPERBOLOID OF ONE SHEET (CONES)

1) PICTURE



2) F CONS?

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & xz \end{vmatrix} = \langle \frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(yz),\ -\frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial z}(xy),\ \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy) \rangle$$

$$= \langle -y, -z, -x \rangle \neq \langle 0,0,0 \rangle$$

3) STOKES' THEOREM

$$\int_C F \cdot dr = \iint_S \text{curl}(F) \cdot d\vec{S}$$

S = INSIDE OF C

4) NOTE SETTING $z = 4$ IN $x^2+y^2-z^2=9 \Rightarrow x^2+y^2-4^2=9 \Rightarrow x^2+y^2=25$

SO C IS A CIRCLE OF RADIUS 5 (AND S IS A DISK OF RADIUS 5)

PARAMETRIZE S $r(x,y) = \langle x, y, 4 \rangle$

$r_x = \langle 1,0,0 \rangle$
$r_y = \langle 0,1,0 \rangle$

$$\hat{N} = r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \underbrace{\langle 0,0,1 \rangle}_{\geq 0}$$

SO
$$\iint_S \text{curl}(F) \cdot d\vec{S} = \iint_D \langle \underbrace{-y}_{-y}, \underbrace{-4}_{-z}, \underbrace{-x}_{-x} \rangle \cdot \underbrace{\langle 0,0,1 \rangle}_{\hat{N}}\, dx\, dy$$

$$= \iint_D -x\, dx\, dy = \int_0^{2\pi}\int_0^5 -5r\cos(\theta)\, r\, dr\, d\theta = \left(\int_0^5 -5r^2\, dr\right)\left(\int_0^{2\pi} \underbrace{\cos(\theta)\, d\theta}_{0}\right) = 0$$

5. *(10 points)* Let D be the parallelogram with vertices $(-2,-3)$, $(0,-1)$, $(1,2)$, $(-1,0)$. Draw a picture of D and calculate

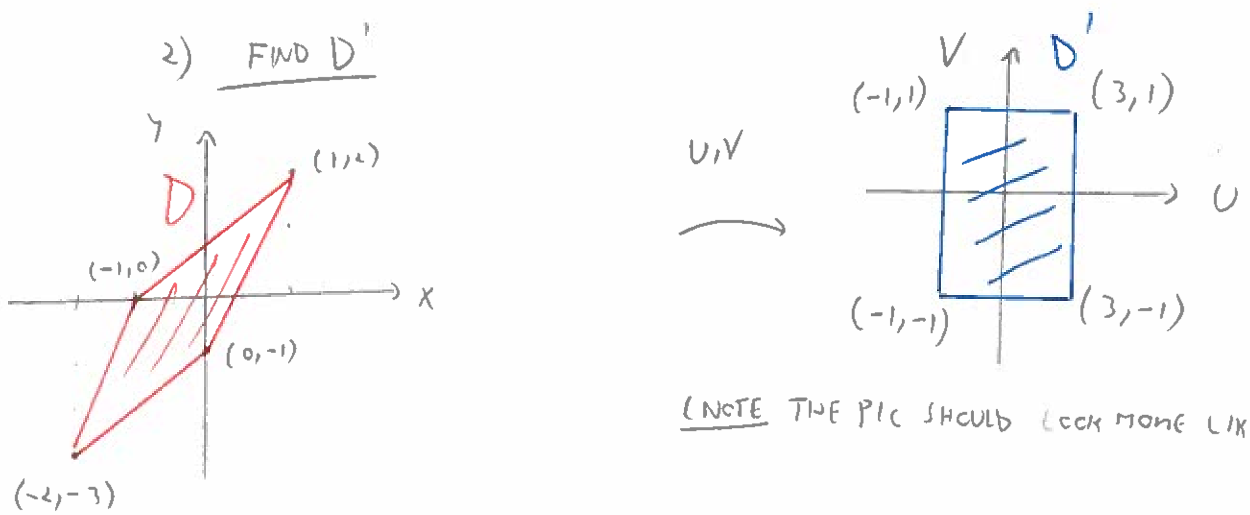
$$\iint_D [redacted]\, dx\,dy$$

$$(y-3x)^2(y-x)^4$$

1) $U = y - 3x$

$V = y - x$

2) FIND D'



(NOTE THE PIC SHOULD LOOK MORE LIKE $\boxed{}$)

$(-2,-3) \rightsquigarrow U = -3-3(-2) = 3$, $V = -3+2 = -1 \rightsquigarrow (3,-1)$

$(0,-1) \rightsquigarrow (-1,-1)$

$(1,2) \rightsquigarrow (-1,1)$

$(-1,0) \rightsquigarrow (3,1)$

3) $dU\,dV = \left|\dfrac{dU\,dV}{dx\,dy}\right| dx\,dy$, $\quad \dfrac{dU\,dV}{dx\,dy} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ -1 & 1 \end{vmatrix} = -2$

$= |-2|\, dx\,dy = 2\, dx\,dy$

$\Rightarrow dx\,dy = \frac{1}{2}\, dU\,dV$

4) $\iint_D (y-3x)^2(y-x)^4\, dx\,dy = \iint_{D'} U^2V^4 \left(\tfrac{1}{2}\right) dU\,dV = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{3} U^2V^4\, dU\,dV$

$= \frac{1}{2}\left(\int_{-1}^{1} V^4\,dV\right)\left(\int_{-1}^{3} U^2\,dU\right) = \cancel{\tfrac{1}{2}}\left(\tfrac{\cancel{2}}{5}\right)\left(\tfrac{1}{3}(3^3+1)\right) = \frac{1}{15}(28) = \frac{28}{15}$

6. *(15 points)* Let S be the part of the cone with parametric equations $\mathbf{r}(u,v) = \langle u\cos(v), u\sin(v), u\rangle$, $0 \le u \le 2$, $0 \le v \le 2\pi$, oriented upwards (no need to draw a picture).

(a) *(5 points)* Find the equation of the tangent plane to S at the point $(\sqrt{3}, 1, 2)$

1) $r_U = \langle \cos(V), \sin(V), 1 \rangle$

$r_V = \langle -U\sin(V), U\cos(V), 0 \rangle$

2) $r_U \times r_V = \begin{vmatrix} i & j & k \\ \cos(V) & \sin(V) & 1 \\ -U\sin(V) & U\cos(V) & 0 \end{vmatrix}$

$= \langle -U\cos(V), -U\sin(V), U\cos^2(V) + U\sin^2(V) \rangle$

$= \langle -U\cos(V), -U\sin(V), U \rangle$

3) FIND U & V

$\langle U\cos(V), U\sin(V), U \rangle = \langle \sqrt{3}, 1, 2 \rangle$

$\Rightarrow U = 2$ AND $U\cos(V) = 2\cos(V) = \sqrt{3} \Rightarrow \cos(V) = \frac{\sqrt{3}}{2}$
$U\sin(V) = 2\sin(V) = 1 \Rightarrow \sin(V) = \frac{1}{2}$ $\Big\}$ $V = \frac{\pi}{6}$

4) Normal Vector $r_U \times r_V = \langle -U\cos(V), -U\sin(V), U \rangle$ $U=2, V=\pi/6$

$= \langle -2\left(\frac{\sqrt{3}}{2}\right), -2\left(\frac{1}{2}\right), 2 \rangle$

$= \langle -\sqrt{3}, -1, 2 \rangle$

5) Point $(\sqrt{3}, 1, 2)$

Normal Vector $(-\sqrt{3}, -1, 2)$

Equation $\boxed{-\sqrt{3}(x-\sqrt{3}) - 1(y-1) + 2(z-2) = 0}$

(Note Version B: $-(x-1) - \sqrt{3}(y-\sqrt{3}) + 2(z-2) = 0$)

(b) *(5 points)* Calculate $\iint_S (x^2+y^2)^2\, dS$, where S is the surface in (a).

1) $r_U \times r_V = \langle -U\cos(V), -U\sin(V), U \rangle$ (From (a))

2) $\|r_U \times r_V\| = \left(U^2\cos^2(V) + U^2\sin^2(V) + U^2\right)^{\frac{1}{2}}$

$= (U^2+U^2)^{\frac{1}{2}}$

$= \sqrt{2U^2}$

$= \sqrt{2}\,U$ 2

3) $$\iint_S (x^2+y^2)^2\, dS = \iint_D f(r(U,V))\, \underbrace{\|r_U \times r_V\|\, dU\,dV}_{dS}$$

$$= \iint_D \left(U^2\cos^2(V) + U^2\sin^2(V)\right)^2 \sqrt{2}\,U\, dU\,dV$$

$$= \int_0^{2\pi}\int_0^2 (U^2)^2 \sqrt{2}\,U\, dU\,dV$$ 1

$$= 2\pi\sqrt{2}\left(\int_0^2 U^5\, dU\right)$$

$$= 2\pi\sqrt{2}\left[\frac{U^6}{6}\right]_0^2$$

$$= \cancel{2}\pi\sqrt{2}\,\frac{2^6}{\cancel{6}\,3}$$

$$= \frac{64\pi\sqrt{2}}{3}$$ 2

(c) *(5 points)* Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle z, y, x \rangle$ and S is the surface in (a).

1) From (a), $r_U \times r_V = \langle -U\cos(V), -U\sin(V), \underbrace{U}_{\geq 0} \rangle$

2) $$\iint_S F \cdot d\vec{S} = \iint_D F(r(U,V)) \cdot (r_U \times r_V)\, dU\,dV$$

$$= \iint_D \langle U, U\sin(V), U\cos(V) \rangle \cdot \langle -U\cos(V), -U\sin(V), U \rangle \, dU\,dV$$

$$= \int_0^{2\pi}\int_0^2 -\cancel{U^2\cos(V)} - U^2\sin^2(V) + \cancel{U^2\cos(V)}\, dU\,dV$$ ┘ 3

$$= -\left(\int_0^2 U^2\,dU\right)\left(\int_0^{2\pi} \sin^2(V)\,dV\right)$$

$$= -\left[\frac{U^3}{3}\right]_0^2 \int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos(2V)\,dV$$

$$= -\frac{8}{3}\left[\frac{V}{2} - \frac{1}{4}\cancel{\sin(2V)}^{\,0}\right]_0^{2\pi}$$

$$= -\frac{8}{3}\left(\frac{2\pi}{2}\right)$$

$$= -\frac{8\pi}{3}$$ ┘ 2

7. *(15 points)*

(a) *(5 points)* Find constants a and b such that $\mathbf{G} = \mathbf{curl(F)}$, where $\mathbf{G} = \langle x^2e^z, 0, -2xe^z\rangle$ and $\mathbf{F} = \langle axye^z, 0, bx^2ye^z\rangle$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axye^z & 0 & bx^2ye^z \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(bx^2ye^z) - \frac{\partial}{\partial z}(0),\ -\frac{\partial}{\partial x}(bx^2ye^z) + \frac{\partial}{\partial z}(axye^z),\ \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(axye^z) \right\rangle$$

$$= \langle bx^2e^z, -2bxye^z + axye^z, -axe^z \rangle$$

WANT

$$= \langle x^2e^z, 0, -2xe^z \rangle$$

3

HENCE $\cancel{b x^2 e^z} = \cancel{x^2 e^z} \Rightarrow \underline{b = 1}$

$-axe^z = -2xe^z \Rightarrow -a = -2 \Rightarrow \underline{a = 2}$

2

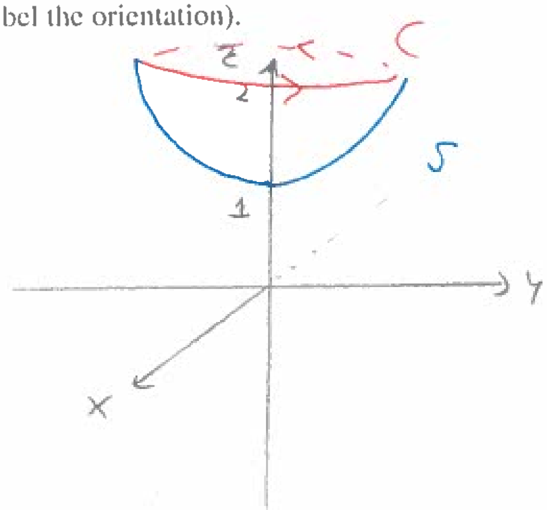
ANSWER $a = 2,\ b = 1$ (VERSION B : $c = 2,\ d = 1$)

HYPERBOLOID OF TWO SHEETS

(TWO CUPS)

(b) *(10 points)* Use your answer in (a) to calculate $\iint_S \mathbf{G} \cdot d\mathbf{S}$, where S is the part of the surface $-x^2 - y^2 + z^2 = 1$ with $1 \leq z < 2$ (without the top) and $\mathbf{G} = \langle x^2 e^z, 0, -2xe^z \rangle$. Assume S is oriented in such a way that the boundary curve C is counterclockwise. Include a picture of S and C (but no need to label the orientation).

1) PICTURE



-1 2

2) $$\iint_S G \cdot d\vec{S} \underset{\text{BY (a)}}{=} \iint_S \text{curl}(F) \cdot d\vec{S} \underset{\text{STOKES}}{=} \int_C F \cdot dr$$

$F = \langle 2xye^z, 0, x^2ye^z \rangle$ ((a) WITH $a=2$, $b=1$) -13

3) PARAMETRIZE C

NOTE $-x^2 - y^2 + \underbrace{z^2}_{z=2} = 1 \Rightarrow -x^2 - y^2 = -3 \Rightarrow x^2 + y^2 = 3$

C : CIRCLE OF RADIUS $\sqrt{3}$ IN THE PLANE $z = 2$, COUNTERCLOCKWISE

$r(t) = \langle \sqrt{3}\cos(t), \sqrt{3}\sin(t), 2 \rangle$ -1 $0 \leq t \leq 2\pi$

$\left[-2\sqrt{3}\sin^3(t)e^2\right]_0^{2\pi} = 0$

$\int_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt$ -1 3

$= \int_0^{2\pi} \langle \underbrace{2(\sqrt{3}\cos(t))(\sqrt{3}\sin(t))e^2}_{2xye^z}, \underbrace{0}_{0}, \underbrace{3\cos^2(t)\sqrt{3}\sin(t)e^2}_{x^2ye^z} \rangle \cdot \langle -\sqrt{3}\sin(t), \sqrt{3}\cos(t), 0 \rangle\, dt$

$= \int_0^{2\pi} -6\sqrt{3}\sin^2(t)\cos(t)e^2\, dt$ -1 2

9. *(15 points)* ...and the Grand Finale!

(a) *(5 points)*

Definition: If $\mathbf{F} = \langle P, Q, R\rangle$ is a vector field and $f = f(x, y, z)$ is a function, then $f\,\mathbf{F}$ is the vector field $\langle fP, fQ, fR\rangle$.

Show that for any vector field $\mathbf{F}$ and any function f,

$$\operatorname{div}(f\,\mathbf{F}) = f\,(\operatorname{div}(\mathbf{F})) + (\nabla f)\cdot \mathbf{F}$$

$\mathrm{DIV}(fF) = \mathrm{DIV}(\langle fP, fQ, fR\rangle)$

$= (fP)_x + (fQ)_y + (fR)_z$

$= f_x P + f P_x + f_y Q + f Q_y + f_z R + f R_z$ 3

$= f P_x + f Q_y + f R_z + f_x P + f_y Q + f_z R$

$= f(P_x + Q_y + R_z) + \langle f_x, f_y, f_z\rangle \cdot \langle P, Q, R\rangle$

$= f\,\mathrm{DIV}(F) + \nabla f \cdot F$ ✓

2

8. *(5 points)* The Pre-Finale...

Let **F** be a vector field, and let S be the sphere centered at $(0,0,0)$ and radius 2, oriented outwards. Calculate $\iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$

SINCE S IS CLOSED, WE CAN USE THE DIVERGENCE THEOREM 2

TO CONCLUDE:

$$\iint_S \text{curl}(F) \cdot d\vec{S} = \iiint_E \underbrace{\text{DIV}(\text{curl}(F))}_{0}\, dx\,dy\,dz = \iiint_E 0 = 0$$

WHERE E IS THE BALL OF RADIUS 2 CENTERED AT $(0,0,0)$ 3

(b) *(10 points)* Suppose that $f = f(x, y, z)$ solves Laplace's equation $\Delta f = 0$ in B (the book writes this as $\nabla^2 f = 0$), where B is the ball of radius R centered at $(0, 0, 0)$. Moreover, suppose that $f = 0$ on the sphere of radius R centered at $(0, 0, 0)$. Use (a) with a special choice of $\mathbf{F}$ to show that $f = 0$ in B.

1) USE (a) WITH $\underline{F = \nabla f}$ TO CONCLUDE:

$$\mathrm{DIV}(f \nabla f) = f\,(\mathrm{DIV}(\nabla f)) + \nabla f \cdot (\nabla f)$$

$$= f(\underbrace{\Delta f}_{0}) + |\nabla f|^2$$

$$\mathrm{DIV}(f\nabla f) = |\nabla f|^2$$

2) NOW INTEGRATE THIS IDENTITY OVER B TO GET:

$$\iiint_B \mathrm{DIV}(f\nabla f) = \iiint_B |\nabla f|^2$$

HOWEVER, BY THE DIVERGENCE THEOREM,

$$\iiint_B \mathrm{DIV}(f\nabla f) = \iint_S (\underset{\overset{\shortparallel}{0}}{f}\nabla f)\cdot d\vec{S}$$, WHERE S IS THE $\underline{\text{SPHERE}}$ OF RADIUS R (ORIENTED OUTWARDS)

$$= 0$$ (SINCE $f = 0$ ON S, BY ASSUMPTION)

THEREFORE $$\iiint_B |\nabla f|^2 = \iiint_B \mathrm{DIV}(f\nabla f) = 0$$, SO $$\iiint_B \underbrace{|\nabla f|^2}_{\geq 0} = 0$$

BUT SINCE $|\nabla f|^2 \geq 0$, THIS IMPLIES $|\nabla f|^2 = 0$, SO $\nabla f = 0$, SO $f = C$ IN B.

3) FINALLY, SINCE f IS CONSTANT $\underline{\text{AND}}$ $f = 0$ IN S, THIS IMPLIES $C = 0$,

AND THEREFORE $f = 0$ IN B.