MAT 267 – FINAL EXAM – SOLUTIONS

1. Multiple Choice

(1) **C**

$$\begin{aligned}&\int_0^3 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt \\ =&\int_0^3 \sqrt{(2)^2 + (3)^2 + (5)^2}dt \\ =&\int_0^3 \sqrt{4+9+25}dt \\ =&\int_0^3 \sqrt{38}dt \\ =&3\sqrt{38}\end{aligned}$$

(2) **D**

$$\begin{aligned}\frac{\partial w}{\partial t} &= \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right) \\ &= \left(x^2+3y^2\right)_x \left(se^{2t}\right)_t + \left(x^2+3y^2\right)_y \left(st^2\right)_t \\ &= (2x)\left(2se^{2t}\right) + (6y)(2st)\end{aligned}$$

(3) **D**

$$\begin{aligned}\nabla f &= \langle f_x, f_y, f_z\rangle \\ &= \left\langle 2xyz - 3z + 4y, x^2z + 4x, x^2y - 3x\right\rangle \\ &= \left\langle 2(1)(0)(1) - 3(1) + 4(0), 1^2(1) + 4(1), 1^2(0) - 3(1)\right\rangle \\ &= \langle -3, 5, -3\rangle\end{aligned}$$

Now normalize $\mathbf{v}$ to get:

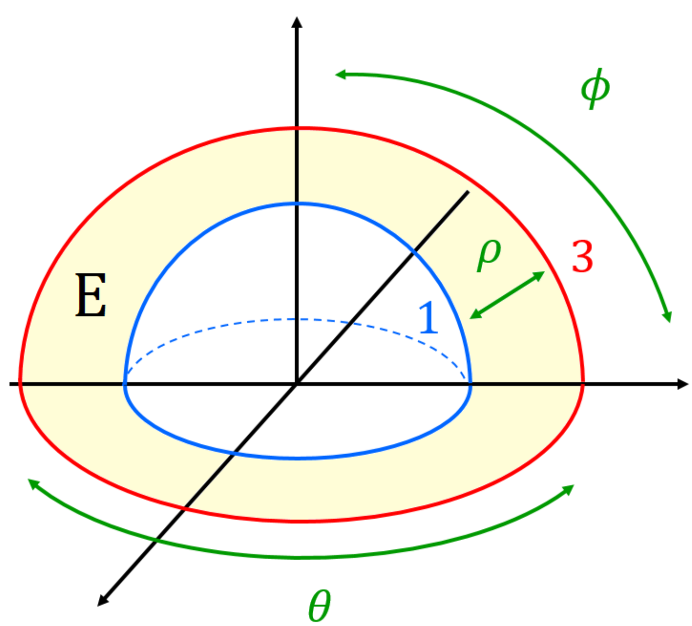
$$\mathbf{v}' = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 3,4,0\rangle}{\sqrt{3^2+4^2+0^2}} = \frac{1}{5}\langle 3,4,0\rangle = \left\langle \frac{3}{5}, \frac{4}{5}, 0\right\rangle$$

And finally:

$$D_{\mathbf{v}}f(1,0,1) = \nabla f(1,0,1)\cdot\mathbf{v}' = \langle -3,5,-3\rangle\cdot\left\langle \frac{3}{5}, \frac{4}{5}, 0\right\rangle = -\frac{9}{5}+\frac{20}{5}+0 = \frac{11}{5}$$

(4) **B**

STEP 1: Picture:



STEP 2: Inequalities

$$z = \sqrt{1 - x^2 - y^2} \Rightarrow x^2 + y^2 + z^2 = 1 \Rightarrow \rho = 1$$
$$z = \sqrt{9 - x^2 - y^2} \Rightarrow x^2 + y^2 + z^2 = 9 \Rightarrow \rho = 3$$

$$\begin{cases} 1 \leq \rho \leq 3 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \frac{\pi}{2} \end{cases}$$

STEP 3: Integrate

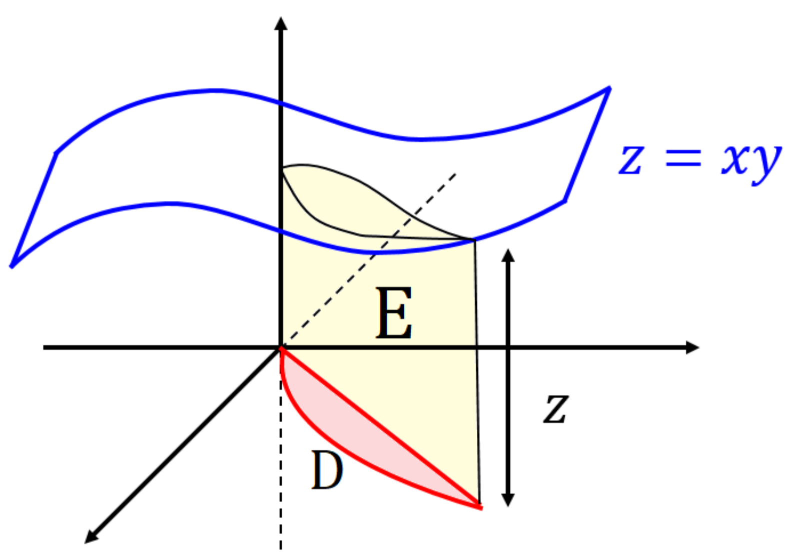
$$\begin{aligned} &\int\int\int_E \sqrt{x^2 + y^2 + z^2}\, dxdydz \\ =&\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_1^3 \rho\, \rho^2 \sin(\phi)\, d\rho d\theta d\phi \\ =&\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^3 \rho^3 \sin(\phi)\, d\rho d\phi d\theta \end{aligned}$$

(5) **A**

$$
\begin{aligned}
&\operatorname{curl}(F)\\
&=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5x-3y & xyz^2 & x^2-z^2 \end{vmatrix}\\
&=\left\langle \frac{\partial}{\partial y}\left(x^2-z^2\right)-\frac{\partial}{\partial z}\left(xyz^2\right), -\frac{\partial}{\partial x}\left(x^2-z^2\right)+\frac{\partial}{\partial z}(5x-3y),\right.\\
&\quad \left.\frac{\partial}{\partial x}\left(xyz^2\right)-\frac{\partial}{\partial y}(5x-3y)\right\rangle\\
&=\left\langle -2xyz, -2x, yz^2+3\right\rangle\\
&=\left\langle -2(1)(2)(1), -2(1), (2)(1)^2+3\right\rangle\\
&=\langle -4, -2, 5\rangle
\end{aligned}
$$

(6) **A**

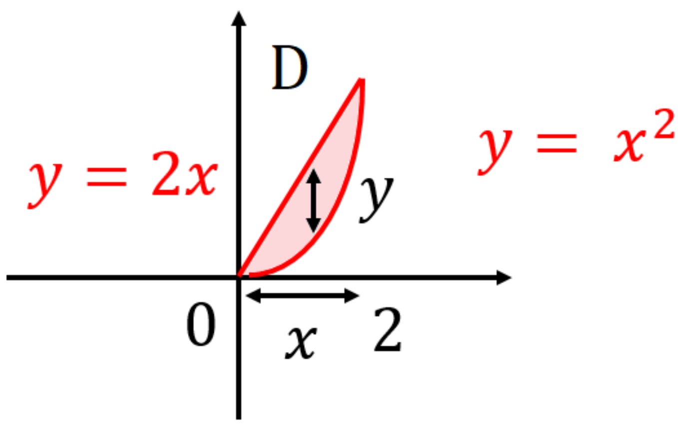
STEP 1: Picture:



STEP 2: Inequalities

$$\text{Small} \leq z \leq \text{Big}$$
$$0 \leq z \leq xy$$

STEP 3: Find D



$$\text{Small} \leq y \leq \text{Big}$$
$$x^2 \leq y \leq 2x$$

Intersection:

$$x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0 \text{ or } x = 2$$

$$0 \leq x \leq 2$$

Therefore we get:

$$\begin{cases} 0 \leq z \leq xy \\ x^2 \leq y \leq 2x \\ 0 \leq x \leq 2 \end{cases}$$

STEP 4: Integrate

$$\begin{aligned}
&\text{Vol }(E) \\
=&\int\int\int_E 1\, dxdydz \\
=&\int_0^2 \int_{x^2}^{2x} \int_0^{xy} 1\, dzdydx \\
=&\int_0^2 \int_{x^2}^{2x} xy\, dydx \\
=&\int_0^2 x \left[\frac{y^2}{2}\right]_{y=x^2}^{y=2x} dx \\
=&\int_0^2 \frac{x}{2}\left((2x)^2 - \left(x^2\right)^2\right) dx \\
=&\int_0^2 \frac{x}{2}\left(4x^2 - x^4\right) dx \\
=&\int_0^2 2x^3 - \frac{x^5}{2} dx \\
=&\left[\frac{x^4}{2} - \frac{x^6}{12}\right]_0^2 \\
=&\frac{2^4}{2} - \frac{2^6}{12}
\end{aligned}$$

$$
\begin{aligned}
&= 8 - \frac{2^4}{3} \\
&= 8 - \frac{16}{3} \\
&= \frac{24}{3} - \frac{16}{3} \\
&= \frac{8}{3}
\end{aligned}
$$

(7) $\boxed{\textbf{B}}$

STEP 1: Slopes

$$
\begin{aligned}
r_u &= \langle 3, 0, 1 \rangle \\
r_v &= \langle -1, 2, 1 \rangle
\end{aligned}
$$

STEP 2: Normal Vector

$$
r_u \times r_v = \begin{vmatrix} i & j & k \\ 3 & 0 & 1 \\ -1 & 2 & 1 \end{vmatrix} = \langle -2, -3 - 1, 6 \rangle = \langle -2, -4, 6 \rangle
$$

STEP 3: dS

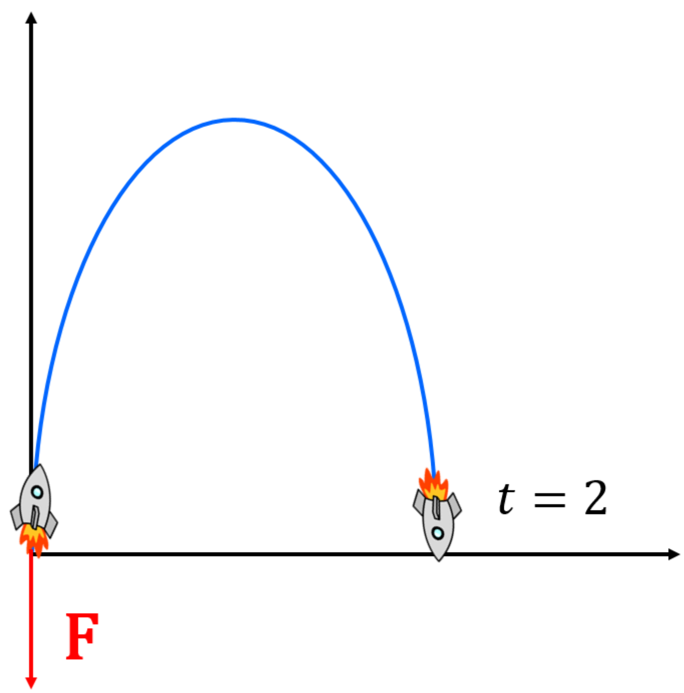
$$
dS = \sqrt{(-2)^2 + (-4)^2 + 6^2} = \sqrt{4 + 16 + 36} = \sqrt{56}
$$

STEP 4: Integrate

$$
\text{Area } (S) = \int \int_S dS = \int_0^4 \int_0^2 \sqrt{56} du dv = 4 \times 2 \times \sqrt{56} = 8\sqrt{56}
$$

2. Free Response

1a.



STEP 1: First let's figure out at which time the speed crashes. For this let $z = 0$:

$$z = 0 \Rightarrow 6t - 3t^2 = 0 \Rightarrow 3t(2-t) = 0 \Rightarrow t = 0 \text{ or } t = 2$$

Hence the rocket crashes at $t = 2$.

STEP 2: Velocity

$$\begin{aligned} r(t) &= \left\langle t^4 + 3t^2, t^2 + 2t, 6t - 3t^2 \right\rangle \\ r'(t) &= \left\langle 4t^3 + 6t, 2t + 2, 6 - 6t \right\rangle \\ r'(2) &= \left\langle 4(2)^3 + 6(2), 2(2) + 2, 6 - 6(2) \right\rangle \\ r'(2) &= \langle 44, 6, -6 \rangle \end{aligned}$$

STEP 3: Speed

$$|r'(2)| = \sqrt{(44)^2 + 6^2 + (-6)^2} = \sqrt{2008} = 2\sqrt{502}$$

1b. First find the acceleration at $t = 0$

$$\begin{aligned} r(t) &= \left\langle t^4 + 3t^2, t^2 + 2t, 6t - 3t^2 \right\rangle \\ r'(t) &= \left\langle 4t^3 + 6t, 2t + 2, 6 - 6t \right\rangle \\ r''(t) &= \left\langle 12t^2 + 6, 2, -6 \right\rangle \end{aligned}$$

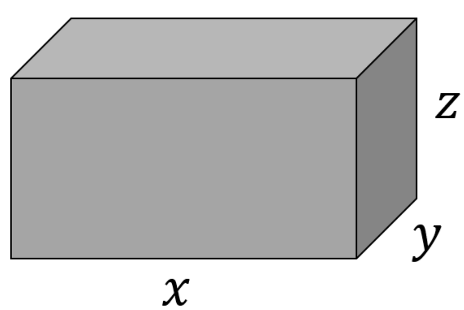
Hence the acceleration at time $t = 0$ is:

$$a(0) = \left\langle 12(0)^2 + 6, 2, -6 \right\rangle = \langle 6, 2, -6 \rangle$$

Therefore, by Newton's Second Law, we have:

$$F(0) = ma(0) = 3\langle 6, 2, -6 \rangle = \langle 18, 6, -18 \rangle$$

2.



STEP 1: Find f

Here the volume is $V = xyz$

But given that $2x+y+z=6$ we get $z = 6-2x-y$ and therefore:

$$f(x,y) = xy(6-2x-y) = 6xy - 2x^2y - xy^2$$

STEP 2: Critical Points:

$$f_x = \left(6xy - 2x^2y - xy^2\right)_x = 6y - 4xy - y^2 = 0 \Rightarrow 6 - 4x - y = 0$$
$$f_y = \left(6xy - 2x^2y - xy^2\right)_y = 6x - 2x^2 - 2xy = 0 \Rightarrow 6 - 2x - 2y = 0$$

(Here we used the fact that $x, y \neq 0$)

The first equation gives us $y = 6 - 4x$ and plugging this into the second equation, we get

$$\begin{aligned} 6 - 2x - 2y &= 0 \\ 6 - 2x - 2(6-4x) &= 0 \\ 6 - 2x - 12 + 8x &= 0 \\ -6 + 6x &= 0 \\ 6x &= 6 \\ x &= 1 \end{aligned}$$

And therefore $y = 6 - 4x = 6 - 4(1) = 2$

Hence the only critical point is $(1, 2)$

STEP 3: Second Derivative Test:

Recall $f_x = 6y - 4xy - y^2$ and $f_y = 6x - 2x^2 - 2xy$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -4y & 6-4x-2y \\ 6-4x-2y & -2x \end{vmatrix}$$

$$D(1,2) = \begin{vmatrix} -4(2) & 6-4(1)-2(2) \\ 6-4(1)-2(2) & -2(1) \end{vmatrix} = \begin{vmatrix} -8 & -2 \\ -2 & -2 \end{vmatrix} = 16-4 = 12 > 0$$

And $f_{xx}(1,2) = -8 < 0$, hence f has a local maximum at $(1,2)$ (which is also an absolute maximum)

STEP 4: Therefore the dimensions that give the largest volume are: $x = 1, y = 2$ and

$$z = 6 - 2x - y = 6 - 2(1) - 2 = 2$$

So the answer is:

$$\begin{cases} x = 1 \\ y = 2 \\ z = 2 \end{cases}$$

3. STEP 1: Parametrize

$$r(x,y) = \langle x, y, x + 3y \rangle$$

STEP 2: Slopes

$$r_x = \langle 1, 0, 1 \rangle$$
$$r_y = \langle 0, 1, 3 \rangle$$

STEP 3: Normal Vector

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{vmatrix} = \langle -1, -3, 1 \rangle$$

STEP 4: dS

$$dS = \|r_x \times r_y\| = \|\langle -1, -3, 1 \rangle\| = \sqrt{1+9+1} = \sqrt{11}$$

STEP 5: Integrate

$$\begin{aligned} &\int\int_S (xy+z)\ dS \\ =&\int_0^2\int_0^1 (xy + x + 3y)\,\sqrt{11}dxdy \\ =&\sqrt{11}\int_0^2 \left[\frac{x^2y}{2} + \frac{x^2}{2} + 3yx\right]_{x=0}^{x=1} \\ =&\sqrt{11}\int_0^2 \frac{y}{2} + \frac{1}{2} + 3ydy \\ =&\sqrt{11}\left[\frac{y^2}{4} + \frac{y}{2} + \frac{3y^2}{2}\right]_0^2 \\ =&\sqrt{11}\left(\frac{4}{4} + \frac{2}{2} + \frac{3(4)}{2}\right) \\ =&\sqrt{11}\,(1+1+6) \\ =&8\sqrt{11} \end{aligned}$$

4. **STEP 1: Parametrize**

$$r(u,v) = \langle 2u, 3v, u+v \rangle$$

STEP 2: Slopes

$$\begin{aligned} r_u &= \langle 2, 0, 1 \rangle \\ r_v &= \langle 0, 3, 1 \rangle \end{aligned}$$

STEP 3: Normal Vector

$$\hat{n} = r_u \times r_v = \begin{vmatrix} i & j & k \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix} = \left\langle -3, -2, \underbrace{6}_{\geq 0} \right\rangle \checkmark$$

STEP 5: Integrate

$$\begin{aligned} &\int\int_S F \cdot d\mathbf{S} \\ &= \int_0^2 \int_0^1 \underbrace{\langle u+v, 2u, 3v \rangle}_{\langle z,x,y \rangle} \cdot \underbrace{\langle -3, -2, 6 \rangle}_{\hat{n}} dudv \\ &= \int_0^2 \int_0^1 -3(u+v) - 2(2u) + 6(3v) dudv \\ &= \int_0^2 \int_0^1 -3u - 3v - 4u + 18v dudv \\ &= \int_0^2 \int_0^1 -7u + 15v dudv \end{aligned}$$

$$
\begin{aligned}
&= \int_0^2 \left[-\frac{7u^2}{2} + 15uv \right]_{u=0}^{u=1} dv \\
&= \int_0^2 -\frac{7}{2} + 15v dv \\
&= \left[-\frac{7v}{2} + \frac{15v^2}{2} \right]_{v=0}^{v=2} \\
&= -\frac{7(2)}{2} + \frac{15(4)}{2} \\
&= -7 + 30 \\
&= 23
\end{aligned}
$$

5. STEP 1: Normal Vector

$$r_\theta \times r_\alpha = a(b + a\cos(\alpha)) \langle \cos(\alpha)\cos(\theta), \cos(\alpha)\sin(\theta), \sin(\alpha) \rangle$$

STEP 2: dS

$$
\begin{aligned}
dS &= \| r_\theta \times r_\alpha \| \\
&= \left\| \underbrace{a(b + a\cos(\alpha))}_{\geq 0} \langle \cos(\alpha)\cos(\theta), \cos(\alpha)\sin(\theta), \sin(\alpha) \rangle \right\| \\
&= a(b + a\cos(\alpha)) \| \langle \cos(\alpha)\cos(\theta), \cos(\alpha)\sin(\theta), \sin(\alpha) \rangle \| \\
&= a(b + a\cos(\alpha)) \sqrt{\cos^2(\alpha)\cos^2(\theta) + \cos^2(\alpha)\sin^2(\theta) + \sin^2(\alpha)} \\
&= a(b + a\cos(\alpha)) \sqrt{\cos^2(\alpha)\left(\cos^2(\theta) + \sin^2(\theta)\right) + \sin^2(\alpha)} \\
&= a(b + a\cos(\alpha)) \sqrt{\cos^2(\alpha) + \sin^2(\alpha)} \\
&= a(b + a\cos(\alpha))
\end{aligned}
$$

STEP 3: Integrate

$$
\begin{aligned}
\text{Area } (S) =& \int_0^{2\pi} \int_0^{2\pi} a(b + a\cos(\alpha)) d\theta d\alpha \\
=& 2\pi \int_0^{2\pi} a(b + a\cos(\alpha)) d\alpha \\
=& 2\pi a \int_0^{2\pi} b + a\cos(\alpha) d\alpha \\
=& 2\pi a \left[b\alpha + a\sin(\alpha) \right]_0^{2\pi} \\
=& 2\pi a \left((b(2\pi)) + a\sin(2\pi) - a\sin(0) \right) \\
=& (2\pi a)(2\pi b) \\
=& 4\pi^2 ab
\end{aligned}
$$

Note: It's for this reason that the donut is sometimes called the product of two circles! In fact its surface area is $(2\pi a) \times (2\pi b)$!