MATH S4062 – FINAL EXAM – SOLUTIONS

1. Let $\epsilon > 0$ be given

   Since $f_n \to f$ a.e. and $m(E) < \infty$, by Egorov's Theorem, there is a closed subset $A_\epsilon \subseteq E$ with $m(E - A_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on A_ϵ.

   Since $f_n \to f$ uniformly, there is N such that if $n > N$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in A_\epsilon$.

   With that same N, if $n > N$ then

$$\begin{aligned}\int_E |f - f_n|\, dx &= \left(\int_{A_\epsilon} + \int_{E - A_\epsilon}\right) |f_n - f| \\ &= \int_{A_\epsilon} \underbrace{|f_n(x) - f(x)|}_{<\epsilon} dx + \int_{E - A_\epsilon} \underbrace{|f_n(x) - f(x)|}_{\leq 2M} dx \\ &\leq m(A_\epsilon)\,\epsilon + 2M\, m(E - A_\epsilon) \\ &\leq m(E)\epsilon + 2M\epsilon \\ &= \epsilon\,(m(E) + 2M)\end{aligned}$$

   Since $\epsilon > 0$ is arbitrary, it follows $\lim_{n\to} \int_E |f - f_n|\, dx = 0$ $\square$

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2.
$$\begin{aligned}\int |f|^p &= \int_{\{x|\ |f(x)|>t\}} |f|^p + \int_{\{x|\ |f(x)|\leq t\}} |f|^p \\ &\geq \int_{\{x|\ |f(x)|>t\}} |f|^p \\ &\geq \int_{\{x|\ |f(x)|>t\}} t^p \\ &= t^p m\left\{x \mid |f(x)| > t\right\}\end{aligned}$$

$$t^p m\left\{x \mid\ |f(x)| > t\right\} \leq \int |f|^p$$

And dividing by $t^p > 0$ gives us the result.

3.

$$F(x,y,z,u,v) = \begin{bmatrix} x^2 - y^2 + 3z - u^3 + v^2 + 4 \\ 2xy + y^2 - 4z - 2u^2 + 3v^4 + 8 \end{bmatrix}$$

To use the Implicit Function Theorem, check that $\det F_{u,v}(2,-1,0,2,1) \neq 0$ (the derivative with respect to what you want to solve for is nonzero)

$$F_{u,v} = \begin{bmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{bmatrix}$$

$$F_{u,v}(2,-1,0,2,1) = \begin{bmatrix} -3(2)^2 & 2(1) \\ -4(2) & 12(1)^3 \end{bmatrix} = \begin{bmatrix} -12 & 2 \\ -8 & 12 \end{bmatrix}$$

$$\det F_{u,v}(2,-1,0,2,1) = (-12)(12) - 2(-8) = -144 + 16 = -128 \neq 0$$

Therefore the Implicit Function Theorem says that there is G such that $(u,v) = G(x,y)$ near $(2,-1,2,1)$. Moreover

$$G'(2,-1) = -\left(F_{u,v}(2,-1,2,1)\right)^{-1}\left(F_{x,y}(2,-1,2,1)\right)$$

$$F_{x,y,z} = \begin{bmatrix} 2x & -2y & 3 \\ 2y & 2x+2y & -4 \end{bmatrix}$$

$$F_{x,y}(2,-1,0,2,1) = \begin{bmatrix} 2(2) & -2(-1) & 3 \\ 2(-1) & 2(2)+2(-1) & -4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 3 \\ -2 & 2 & -4 \end{bmatrix}$$

$$G'(2,-1,0) = -\begin{bmatrix} -12 & 2 \\ -8 & 12 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 2 & 3 \\ -2 & 2 & -4 \end{bmatrix} = -\left(-\frac{1}{128}\right)\begin{bmatrix} 12 & -2 \\ 8 & -12 \end{bmatrix}\begin{bmatrix} 4 & 2 & 3 \\ -2 & 2 & -4 \end{bmatrix}$$

$$= \frac{1}{128}\begin{bmatrix} 52 & 20 & 44 \\ 56 & -8 & 72 \end{bmatrix} = \frac{1}{32}\begin{bmatrix} 13 & 5 & 11 \\ 14 & -2 & 18 \end{bmatrix}$$

4. **STEP 1:** Using the first property, we get

$$\begin{aligned}(f \star K_n)(x) - f(x) =& \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)K_n(y)dy - f(x)\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)dy \\ =& \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x-y) - f(x)\right)K_n(y)dy\end{aligned}$$

And therefore, we have for every $\delta > 0$

$$\begin{aligned}\left|(f \star K_n)(x) - f(x)\right| =& \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x-y) - f(x)\right)K_n(y)dy\right| \\ \leq& \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(x-y) - f(x)\right|\left|K_n(y)\right|dy \\ =& \frac{1}{2\pi}\int_{|y|<\delta} \left|f(x-y) - f(x)\right|\left|K_n(y)\right|dy \\ &+ \frac{1}{2\pi}\int_{|y|\geq\delta} \left|f(x-y) - f(x)\right|\left|K_n(y)\right|dy\end{aligned}$$

STEP 2: Let $\epsilon > 0$ be given.

Then since f is continuous at x, there is a $\delta > 0$ such that if $|y| < \delta$, then $|f(x-y) - f(x)| < \frac{\pi\epsilon}{M}$. In that case, using the second property, we get

$$\begin{aligned}&\frac{1}{2\pi}\int_{|y|<\delta} \underbrace{\left|f(x-y) - f(x)\right|}_{<\frac{\pi\epsilon}{M}}\left|K_n(y)\right|dy \\ <& \left(\frac{\pi\epsilon}{2\pi M}\right)\int_{|y|<\delta}\left|K_n(y)\right|dy \\ \leq& \left(\frac{\epsilon}{2M}\right)\int_{-\pi}^{\pi}\left|K_n(y)\right|dy \\ \leq& \left(\frac{\epsilon}{2M}\right)(M) \\ =& \frac{\epsilon}{2} \checkmark\end{aligned}$$

STEP 3: Let $C = \sup_y |f(y)|$

By the third property, there is $N > 0$ such that for all $n \geq N$ and all x we have

$$\int_{|y|\geq\delta} |K_n(y)|\, dy < \frac{\pi\epsilon}{2C}$$

This implies that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{|y|\geq\delta} |f(x-y) - f(x)|\, |K_n(y)|\, dy \\
\leq &\frac{1}{2\pi}\int_{|y|\geq\delta} \underbrace{|f(x-y)| + |f(x)|}_{\leq 2C} |K_n(y)|\, dy \\
= &\frac{2C}{2\pi}\int_{|y|\geq\delta} |K_n(y)|\, dy \\
< &\frac{C}{\pi}\left(\frac{\pi\epsilon}{2C}\right) = \frac{\epsilon}{2}\checkmark
\end{aligned}$$

STEP 4: Putting everything together, with N as above, if $n > N$ then

$$|(f \star K_n)(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\checkmark$$

And therefore $\lim_{n\to\infty}(f \star K_n)(x) = f(x)$ □

5. Let $f(x) = e^x$ and $\epsilon = 1$, and suppose that any polynomial p, and for all $x \in \mathbb{R}$ we have $|f(x) - p(x)| < 1$

Notice in particular that

$$|f(x)| = |f(x) - p(x) + p(x)| \leq |f(x) - p(x)| + |p(x)| < 1 + |p(x)|$$

Since p is a polynomial, we have $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, hence $|f(x)| < 1 + |p(x)|$ implies

$$0 \leq e^x \leq 1 + |a_0| + |a_1| |x| + \cdots + |a_n| |x|^n$$

Dividing by $|x|^n$ we get

$$0 \leq \frac{e^x}{|x|^n} \leq \frac{1}{|x|^n} + \frac{|a_0|}{|x|^n} + \frac{|a_1|}{|x|^{n-1}} + \cdots + |a_n|$$

But now, letting $x \to \infty$, the right-hand-side tends to $|a_n|$, so for large x, the right-hand-side is bounded, but this contradicts

$$\lim_{x \to \infty} \frac{e^x}{|x|^n} = \infty$$

Which you obtain after repeated applications of L'Hôpital's rule. $\Rightarrow\Leftarrow$ $\square$