MATH 409 – FINAL EXAM – STUDY GUIDE

The final exam will take place on **Wednesday, December 15, 2021** from 8 am to 10 am in our usual lecture room. It is a closed book and closed notes exam, and counts for 35% of your grade. It will be an in-person exam and **NO** books, notes, calculators, cheat sheets will be allowed. Please bring your student ID card (or other government ID), for verification purposes.

There will be 7 questions in total: 1 from Midterm 1 material, 2 from Midterm 2 material, and 4 from the material after Midterm 2. The format is the same as usual: There will be a mix of definitions, counterexamples, computational problems, problems similar to the homework and one of the "Proofs you should know" below.

This is the study guide for the exam, and is just meant to be a *guide* to help you study, just so we're on the same place in terms of expectations. For a more thorough study experience, look at all the lecture notes, homework, and practice exams.

This exam covers the whole course, from sections $4-34$ inclusive, with the exception of sections 6, 13, 16, $21-27$, and 31.

Important: This study guide starts from section 19, but please **also** look at the study guides for midterms 1 and 2, since the final is cumulative. The only thing you can sort of ignore are sections $1-3$, since they are more of an introduction to the course, the only thing from those 3 sections that is important is the triangle inequality and how to use induction.

Date: Wednesday, December 15, 2021.

Proofs you should know

Note: You **only** need to know the proofs from the list below:

(1) Archimedean Property

(2) The Squeeze Theorem (both for sequences and for functions)

(3) Monotone Sequence Theorem

(4) $\liminf s_n = -(\limsup -s_n)$

(5) (s_n) converges $\Rightarrow$ (s_n) Cauchy

(6) Divergence Test and Comparison Test

(7) f is continuous implies f is bounded

(8) Product Rule and Quotient Rule

(9) f monotone $\Rightarrow$ f integrable, and f continuous $\Rightarrow$ f integrable

(10) Fundamental Theorem of Calculus 1

(11) Fundamental Theorem of Calculus 2

Definitions you should know

Please also look at the definitions from the midterm 1 and 2 study guides, they are also fair game for the exam.

(1) f is uniformly continuous on $[a, b]$

(2) $\lim_{x\to a} f(x) = L$, and all the variations thereof, such as $\lim_{x\to a^+} f(x) = L$ (or same with a^-) or $\lim_{x\to a} f(x) = \infty$ of $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = \infty$, and same thing with $-\infty$

(3) $f'(a)$, f differentiable at a

(4) Product Rule, Quotient Rule, Chain Rule

(5) Fermat's Theorem, Rolle's Theorem

(6) Mean Value Theorem

(7) Inverse Function Theorem (in the L'Hôpital Notes)

(8) L'Hôpital's Rule

(9) Partition P, $M(f, [t_{k-1,t_k}])$ and $m(f, [t_{k-1}, t_k])$, $U(f, P)$ and $L(f, P)$, $U(f)$ and $L(f)$

(10) f is integrable on $[a, b]$, Darboux integral $\int_a^b f(x)dx$

(11) Riemann Sum

(12) Cauchy Criterion for integrability

(13) MVT for Integrals

(14) The Fundamental Theorem of Calculus (both versions)

Note: For a nice review video of the topics in Chapters 1−3, check out this video: Essence of Analysis (ignore the topology part)

Section 19: Uniform Continuity

- Know the definition of uniform continuity, and understand how it's different from the definition of continuity. The main point is that, in uniform continuity, the δ does not depend on x and y, it's independent of where you are

- Use the $\epsilon - \delta$ definition of uniform continuity to show that f is uniformly continuous. For example, show that $f(x) = x^2$ is uniformly continuous on $[-1, 3]$ (Example 1) or that $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[2, \infty)$ (Example 2)

- I won't ask you to show that f is not uniformly continuous

- If you want some practice problems, check out 19.1(b) and 19.2

- Show that if f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$. Again, the usual Bolzano-Weierstraß trick. Use this, for example, to show that $f(x) = x^2$ is uniformly continuous on $[-1, 3]$

- Show that if s_n is Cauchy and f is uniformly continuous, then $f(s_n)$ is Cauchy. Use this to show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$

- Know what a continuous extension is

- Know that if $f : (a, b) \to \mathbb{R}$ has a continuous extension $\tilde{f} : [a, b] \to \mathbb{R}$, then f is uniformly continuous on (a, b). Use this to show that $f(x) = x \sin\left(\frac{1}{x}\right)$ is uniformly continuous on $(0, 1]$

- Know that if $f : (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b), then f has a continuous extension $\tilde{f} : [a, b] \to \mathbb{R}$. Use this to show that $f(x) = \sin\left(\frac{1}{x}\right)$ is not uniformly continuous on $(0, 1]$

- Know that if $f : [a, b] \to \mathbb{R}$ and if f' is bounded on (a, b), then f is uniformly continuous, and check out the proof. Use this to show that $f(x) = \frac{1}{x}$ is uniformly continuous on $[2, \infty)$.
- In the book, you can check out Example 1 (which is just practice with continuity), but ignore Discussion 19.3
- Problems 19.2, 19.4 – 19.6 and 19.8 are good practice with uniform continuity

Section 20: Limits of Functions

- Know the $\epsilon - \delta$ definition of $\lim_{x\to a} f(x) = L$. I could ask you variations thereof, like $\lim_{x\to a^+} f(x) = L$ (or same with a^-) or $\lim_{x\to a} f(x) = \infty$ of $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = \infty$, and same thing with $-\infty$
- **Use the above definition to evaluate a limit,** good examples can be found in the videos below, as well as in AP1 on the section 20 homework
  - Example 1
  - Example 2
  - Example 3
  - Example 4
  - Example 5
  - Example 6
  - Example 7
- Prove the squeeze theorem for limits (AP2 on the section 20 Homework and this video)

- You can ignore the problem with $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$
- Know how to prove some limit laws, such as $\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$ (this is the same as usual), also see 20.20(a)(b)
- Ignore the "Chen Lu Fact" in the section 20 lecture
- Check out the proof of the "Two Sides Fact" in the section 20 lecture, especially the ($\Leftarrow$) part, it's a typical $\epsilon - \delta$ proof.

Section 28: Basic Properties of the Derivative

- Define $f'(a)$, f is differentiable at a
- Calculate basic derivatives, such as $(x^n)'$ or $(\sin(x))'$, also check out 28.2
- Show $|x|$ is not differentiable at 0
- Know differentiable $\Rightarrow$ continuous, and check out its proof, it's very elegant
- Prove $(f+g)' = f' + g'$ and $(cf)' = cf'$
- Prove the product rule and the quotient rule. See this video for a proof of the product rule. For the quotient rule, you can use the cool Peyam way, or the boring book way.
- Ignore the proof of the Chen Lu
- It's useful to know in practice that $\frac{f(x+h)-f(x)}{h} = f'(x) + O(h)$

- Ignore the problem with $f(x) = e^{-\frac{1}{x}}$ on the section 28 homework.
- Ignore 28.15
- It's useful to look at 28.6, since it's a classical exercise

Section 29: The Mean Value Theorem

- State Fermat's Theorem and understand it's proof (it should be graphically intuitive), I think the proof in the notes makes more sense than the one in the book
- State Rolle's Theorem and understand how it follows from Fermat's Theorem, check out this video if interested
- State the Mean Value Theorem. Understand how Rolle is a special case of it. You don't need to know the function that's used here, but please understand how Rolle is used in this proof
- Know the 3 applications of the MVT in the lecture notes, they are good applications. Here is a video on the application to fixed points: MVT and Fixed Points
- Check out the section on the IVT for derivatives, but you definitely don't need to know the proof.
- Know the Inverse Function Theorem (see the notes on L'Hôpital's Rule). Honestly, here I mainly care about the formula and about how to use it (see Examples 1 and 2), you don't need to know the proof
- Ignore the AP with $f'(x) = f(x)$ in the section 29 homework

- Definitely check out 29.4 and 29.5, those are classical exercises, and also check out 29.18, it's an excellent exercise with Cauchy sequences, here's a link to a (similar) solution: Banach Fixed Point Theorem (here I use $d(x, y)$ instead of $|x - y|$, but the proof is the same)

Section 30: L'Hôpital's Rule

- Know the precise statement of L'Hôpital's rule, with all the assumptions (the one in the notes, not the book), but of course you can ignore the proof
- Ignore the Generalized Mean Value Theorem and its proof
- Check out 30.6 (Solutions) and the AP problem with L'Hôpital's rule, but ignore 30.7

Section 32: The Riemann Integral

- Know all the terminology related to integration: Partition P, $M(f, [t_{k-1,t_k}])$ and $m(f, [t_{k-1}, t_k])$, $U(f, P)$ and $L(f, P)$, $U(f)$ and $L(f)$
- Define f is integrable on $[a, b]$ and $\int_a^b f(x)dx$
- There are really two types of problems that you can have with integration: Either ones where you start with P and then find $M(f, [t_{k-1}, t_k])$ and build your way up to $\int_a^b f(x)dx$, or problems where you use the Cauchy criterion for integrability (see below)

- Show that a function like $f(x) = x^2$ is integrable on an interval, see Example 2 in the notes or 32.1. In that case I would give you the formula for $\sum_{k=1}^{n} k^2$
- Show that a function is not integrable, such as the one in the notes. Usually in that case it's enough to show that $L(f) \neq U(f)$
- Although I won't ask you to reprove that $L(f) \leq U(f)$, you should still understand the main ideas. In particular, understand why adding an extra point to P decreases $U(f, P)$ (the picture on page 10 of the Lecture 24 notes is useful here), and understand the trick with $P \cup Q$ in Lemma 2, it appears a lot here, and finally, understand the idea of taking inf over Q and sup over P on the last page of the Lecture 24 notes
- For Riemann Integrals, understand how Riemann Integration is different from Darboux Integration. You don't need to know the thing about mesh, and you don't really need to know the definition of Riemann Integrable
- Know the fact that Darboux Integrable is equivalent to Riemann Integrable, but ignore its proof
- Know the definition of the Cauchy criterion, but ignore its proof.
- The Cauchy criterion is a **very** useful way of showing that f is integrable
- Ignore the discussion on the Riemann Lebesgue Theorem
- Ignore the APs in the section 32 homework

Section 33: Properties of the Integral

- Know how to prove the two applications: f monotone $\Rightarrow f$ integrable, and f continuous $\Rightarrow f$ integrable
- Know that f and g integrable $\Rightarrow f + g$ and cf are integrable. You don't need to know the proof, but I do want you to notice what role the Cauchy criterion plays here.
- Know the 4 facts in the section "More Properties" in the lecture notes. I'd say the proofs of Facts 1 and 2 are important, they're good exercises with integration and $\epsilon - \delta$
- Know the MVT for integrals, and notice how the proof in the notes follows from the Mean Value Theorem. There are some fun problems with the MVT for integrals in the practice final and the review session notes
- Ignore the section on Integrals and Limits
- Check out Problems 33.4, 33.7, and 33.10, they are very classical, but ignore all the APs

Section 34: The Fundamental Theorem of Calculus

- What can I say, it's the most important theorem in the course ☺
- Know the statement of FTC 1 and FTC 2 and their proofs. You can use the proofs in the book or in my notes, although I'd say the ones in my notes are hopefully a bit easier to follow. Also check out the following videos: FTC 2 Proof and FTC 1 Proof

- Ignore the section on FTC 1+
- Know the statements of Integration by Parts and u-substitution, and notice how they just follow from the product rule and the chain rule, although I won't ask for a proof. Hopefully you've used them before in your calculus lives!
- I won't really ask about the two fun examples at the end of the notes, but they're nice to look at. But please check out 34.10 though.