FINAL EXAM – STUDY GUIDE

The Final Exam takes place at the following times:

(1) Section 501 (MWF 12:40-1:30): Monday, December 13, 10:30-12:30 PM, 111 Heldenfels

(2) Section 512 (MWF 1:50-2:40), Tuesday, December 14, 3:30-5:30 PM, 111 Heldenfels

It will be an in-person exam and **NO** books/notes/calculators/cheat sheets will be allowed. Please bring your student ID card (or other government ID), for verification purposes. The final counts for 20 % of your grade, and covers **only** Chapter 16. **The final exam is not cumulative**

This is the study guide for the exam, and is just meant to be a *guide* to help you study, just so we're on the same place in terms of expectations. For a more thorough study experience, look at all the lectures, quizzes, practice exams, and Webassign. The length and difficulty of the exam will be close to the Peyam $1-4$ exams and the problems in the book. Beware that the Webassign problems are a bit too easy. There will be *tentatively* be 8 problems in total. There should not be any surprises in terms what to expect. As usual, know the 8 surfaces from the 14.1 lecture.

YouTube playlist: Check out the following YouTube Playlist from the Chapter 16 material

Useful trig identities to know:

(1) $\sin^2(x) + \cos^2(x) = 1$

(2) $1 + \tan^2(x) = \sec^2(x)$

(3) $\cos(-x) = \cos(x), \sin(-x) = -\sin(x)$

(4) $\sin(2x) = 2\sin(x)\cos(x)$, $\cos(2x) = \cos^2(x) - \sin^2(x)$

(5) $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$, $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$

Useful integration techniques to know:

(1) $u-$substitution

(2) Integration by parts, but I'll only ask you about easy cases like $\int xe^x dx$

(3) $\int \cos^2(x)dx, \int \sin^2(x)dx$; check out this video in case you forgot

(4) Trig integrals like $\int \cos^3(x)\sin(x)dx$

(5) $\int \cos^3(x)dx$ and $\int \sin^3(x)dx$

(6) **No** trig sub and **No** partial fractions on this exam

Note: The problems below refer to the problems in the book

Section 16.1: Vector Fields

- I won't ask you anything about this section
- Here's a great overview of all the topics in chapter 16: Vector Calculus Overview

Section 16.2: Line Integrals

- Remember the three important parametrizations in the Line Integral lecture: The Circle, the Line Segment, and Graphs of Functions, see Parametric Equations
- Calculate $\int_C f(x,y)ds$ or $\int_C f(x,y,z)ds$, like problems 3, 4, 9 $-$ 12. And remember that it represents the area under a fence. Check out the following videos for examples: Example 1, Example 2, Example 3, Example 4
- The easiest way to memorize this is by using $ds = \sqrt{(dx)^2 + (dy)^2}$ (in the two-dimensional case), see Line Integral Derivation
- Calculate $\int_C Pdx + Qdy$ or $\int_C Pdx + Qdy + Rdz$, like problems 5, 6, 8, 13 $-$16. Check out this video for an example.
- The easiest way to memorize this is by using $dx = \frac{dx}{dt}dt = x'(t)dt$
- You don't need to know the interpretation given in the Line Integral lecture with the shadows
- Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, like problems 19-22. Check out this video for an example.
- Find the work done by a vector field $\mathbf{F}$ on a curve C, like 40 $-$ 41
- Also check out Problem 50

Section 16.3: The Fundamental Theorem for Line Integrals

- Know that in $2D$, $\mathbf{F} = \langle P, Q \rangle$ is conservative if and only if $P_y = Q_x$. (Peyam is Quixotic), like in problems 3-10

- (16.5) Know that in $3D$, $\mathbf{F} = \langle P, Q, R \rangle$ is conservative if and only if $\operatorname{curl} \mathbf{F} = \langle 0, 0, 0 \rangle$.
- Find a function f such that $\mathbf{F} = \nabla f$, like in $12 - 18$. This works in any dimensions. You can use the method in lecture or in the book, whichever you prefer.
- Use the FTC for line integrals to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, like in $12 - 18$. Check out this video for an example. Both the 2D case and the 3D case are fair game for the exam.
- Know the neat fact that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C if and only if $\mathbf{F}$ is conservative. You don't need to know their proofs. See also problems 19 and 20.
- If there is a hole, I will explicitly mention that there is one. Check out 35 for an interesting example
- The problems in the FTC lecture are good problems as well. See for instance the following videos: FTC Example, FTC 3D Example

Section 16.4: Green's Theorem

- Use Green's theorem to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$. Remember that this only works for closed curves C. See the problems in the Green's Theorem lecture, as well as Problems 1-14. Check out this video and this video for an example.
- Beware about orientation: Make sure that if you walk along the curve, your region is to your **left**. Think WA**L**K **L**EFT.
- Use Green's theorem to find the area enclosed by a curve. The easiest way to remember this is:

$$A = \frac{1}{2}\int_C \begin{vmatrix} x & y \\ dx & dy \end{vmatrix} = \frac{1}{2}\int_C xdy - ydx$$

For practice, look at the example in lecture, Problem 21, and this video: Area of Ellipse

- You don't need to memorize the formula for the area of a polygon (problem 21), but you need to know how to derive it, see this video: Area of Polygon

Section 16.5: Curl (and Divergence)

- Find the curl of a vector field. Problems $1-8$ are good practice. The easiest way to memorize it is just by remembering that $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$

- Just know that intuitively, the curl represents the rotation of a vector field; you don't need to know the detailed description that I gave in lecture.

- Know that $\mathbf{F}$ is conservative if and only if $\text{curl}(\mathbf{F}) = \langle 0, 0, 0\rangle$. See Problems $13 - 18$. I might combine that with a question about the FTC for line integrals, like in the lecture on Curl and Divergence. or the mock final, so at this point it might be good practice to go back to 16.3 and do Problems $15-18$. In such a problem, you **HAVE** to check the curl, it's not enough on the exam just to find f such that $\mathbf{F} = \nabla f$

- Know that $\text{curl}(\nabla f) = \langle 0, 0, 0\rangle$ and $\text{div}(\text{curl}(\mathbf{F})) = 0$; The way to remember this is that if you apply a new topic in the book to the topic right before that, you should get 0. For example, in the book, div comes after curl, that's why $\text{div}(\text{curl}(\mathbf{F})) = 0$.

- Use the preceding fact to show that a vector field cannot be written as a curl, see Problems 19 and 20.
- You might also want to check out Problems $23 - 29$
- Ignore 35 in 16.5

Section 16.6: Parametric Surfaces and their Areas

- Know how to parametrize familiar surfaces, like the cylinder, spheres, planes, and functions. See the examples at the end of the Parametric Surfaces Lecture notes as well as problems 20, 23, 24, 26. Here is an overview of how to do that: Parametric Surfaces
- Ignore the example at the end of the lecture notes with solids of revolution
- Sometimes the parametrization will be given (like the helicoid), but sometimes you have to figure it out on your own.
- Find the equation of the tangent plane to a surface, see problems $33 - 36$. Check out this video for an example.
- Know that $\hat{n} = r_u \times r_v$ represents the normal vector to your surface, and $dS = \|r_u \times r_v\| \, dudv$ represents the area of a tiny parallelogram.
- Find surface areas of parametric curves; the examples the surface area lecture are good examples. Problems $39 - 48$ are all good practice questions. Check out this video for an example.
- The easiest way to remember $\int\int_D \|r_u \times r_v\| \, dudv$ is by writing the area as $\int\int_S 1 dS$ and using the definition of dS.

- **DON'T** memorize the formula for the surface area of the graph of a function (Formula 9 in section 16.6), it's *much* easier to derive by using the parametrization $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$

Section 16.7: Surface Integrals

- Find surface integrals of functions. The easiest way to remember the formula is by starting with $\int\int_S f dS$ and by using the formula $dS = \|r_u \times r_v\| \, dudv$. Again, think height times base, where your height is f and your base is dS (the mini-parallelogram)

- Problems 5−20 and the problems in the Surface Integral Lecture are all good practice questions. Check out this video for an example

- Again, **DON'T** memorize formula 4 in section 16.7, it's much easier just to use the parametrization $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$ and to use the definition of the surface integral

- Find surface integrals of vector fields. The easiest way to memorize it is by writing it as $\int\int_D \mathbf{F}\cdot\hat{n} \, dudv$ and by using $\hat{n} = \mathbf{r}_u \times \mathbf{r}_v$.

- In this case, you're dotting $\mathbf{F}$ with the normal vector, whereas for line integrals you're dotting $\mathbf{F}$ with the direction vector! That's because the direction vector is to a line what the normal vector is to a curve

- The problems in the surface integral lecture are good practice problems, as well as Problems 21 − 32. Check out this video for an example

- **PLEASE DON'T** memorize Formula 10 in section 16.7, I'm begging you!!! Again, it's so much easier to use $\mathbf{r}(x, y) = \langle x, y, g(x, y)\rangle$

- **For surface integrals, orientation matters!!!! Unless otherwise specified, make sure that your normal vector points upwards (for surfaces) or outwards (for closed surfaces like the sphere)**. Most of the time this just amounts to checking that the third component of $\hat{\mathbf{n}}$ is ≥ 0, but a picture should really help.

- Know the "adult" definition of the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{n} = \frac{\hat{\mathbf{n}}}{\|\hat{\mathbf{n}}\|}$ is the **unit** normal vector to S (see end of surface integral lecture). I would give you the formula if needed.

Section 16.9: The Divergence Theorem

- (16.5) Find the Divergence of a vector field. Problems $1-8$ in 16.5 are good practice. While you're at it, you might also want to check out Problem 25 in 16.5.

- Use the divergence theorem to calculate surface integrals. The problems in the Divergence Theorem lecture, as well as the problems $5-12$ in 16.9 are good practice. Check out this and this video for an example

- **Caution:** The divergence theorem only works for **closed** surfaces. If S is not closed, then make sure to close it! See the last example in the divergence theorem lecture, as well as problems 17 and 18 and the problem on the mock final to see how to deal with open surfaces.

- **Remember that for orientation, *outward* orientation takes precedence over *upward* orientation.**

Section 16.8: Stokes' Theorem

- There are two ways to use Stokes' theorem:

(1) Use Stokes to calculate $\int\int_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$. This just amounts to calculating $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the boundary of S. See Problem 2 and the problems in the Stokes' Theorem lecture. Problems 3-5 are a bit too complicated, everything will most likely be oriented in the positive $z-$axis.

- **Note: MOST** of the time, C will be oriented counterclockwise, but remember that the best way to check the orientation is that if you walk on the curve C with your head in the direction of the normal vector $\hat{\mathbf{n}}$, then the surface S needs to be on the **LEFT** (wa**L**k left).

- To practice with orientation, it's useful to check out the following video: Integral over a barrel

(2) Use Stokes to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$. This just amount to calculating $\int\int_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$, where S is (usually) the interior of C. See Problems $7-10$ as well as the problems in the Stokes' theorem video for practice.

- Check out the following videos for Stokes examples: Stokes Theorem 1, Stokes Theorem 2, Stokes Theorem 3

Section 16.10: Summary

- Check out the summary in section 16.10
- One of the hard things to know is when to use which FTC. This is the point of the final exam review session.
- Please also look at the handouts on my website which explain the similarities between the FTC and which give a roadmap of which theorem to use when. Notice that all the FTCs say that

the integral of a derivative equals to the function itself, or the integral on the boundary of that function.