MATH S4062 – FINAL EXAM STUDY GUIDE

The Final Exam will take place on **Friday, August 12, 2022** from 10:45 am to 12:20 pm in our usual lecture room. It is a closed book and closed notes exam, and counts for 40% of your grade.

This exam covers the whole course. This corresponds to Chapters 7, 8, 9, and 11 of Rudin, and the material from Homework 1-9.

There will be 5 questions in total, so expect to spend roughly 20 mins per question. At least 3 questions will come from chapters 9 and 11.

(1) The first one will be one of the "Proofs you should know" that will be found in the study guide.

(2) One question will be taken directly from Homework 1-9

(3) One question will be taken from the Review Session

(4) One new question

(5) Another new question

I will probably not ask for definitions, but you can always include them if you want some partial credit.

Date: Friday, August 12, 2022.

Proofs you should know

Know how to prove the following theorems. I will **NOT** ask you to restate the theorems:

(1) If $f_n \to f$ uniformly and each f_n is continuous at x_0 then f is continuous at x_0

(2) The Power Series converges uniformly on $[-r, r]$ if $r < R$

(3) $\widehat{f \star g}\,(n) = \hat{f}(n)\hat{g}(n)$

(4) The Chain Rule (the proof I gave during lecture)

(5) Implicit Function Theorem but **ONLY** **STEPS 1, 2** (up to "f invertible"), and just the differentiation formula in **STEP 5**.

(6) Countable sub-additivity (Property 2 in Lecture 18)

(7) Bounded Convergence Theorem

(8) The two corollaries about convergence of series from Lecture 22

(9) The two regularity facts from Lecture 22

Definitions you should know

Although I will not explicitly ask for definitions, it's still useful to know the concepts below

(1) $f_n \to f$ pointwise

(2) $f_n \to f$ uniformly

(3) A sequence of functions that converges pointwise but not uniformly ($f_n = x^n$ on $[0,1]$ works)

(4) The $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ example, whose uniform limit is not differentiable

(5) $C[a,b]$ and its metric d

(6) $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly

(7) (f_n) is bounded

(8) (f_n) is equicontinuous

(9) Arzelà-Ascoli Theorem

(10) Fixed points, Contraction

(11) Weierstraß Approximation Theorem

(12) Power series, Radius of Convergence

(13) Fubini for series, including the counterexample

(14) Fourier series on $[-\pi,\pi]$, $\hat{f}(n)$ on $[-\pi,\pi]$, $S_N(f)$

(15) $f \star g$, both on $[-\pi,\pi]$ and on $(-\infty,\infty)$

(16) Self-Adjointness of the Fourier transform

(17) $\|A\|$

(18) An unbounded linear transformation (see Homework 6)

(19) f differentiable at x and f differentiable

(20) $\frac{\partial f_i}{\partial x_j}$, $[f'(x)]$, $D_u f$, ∇f

(21) Mean Value Theorem and the counterexample preceding it

(22) Inverse Function Theorem, diffeomorphism

(23) Implicit Function Theorem: Just know the condition det $F_y(x_0, y_0)$ and the formula for $G'(x_0)$, the rest is not too important. Remember: "the derivative with respect to what you want to solve for is nonzero"

(24) Clairaut's Theorem

(25) $m_\star(E)$ and its properties

(26) Measurable sets and their properties, $m(E)$

(27) Measurable Function

(28) Characteristic Function, Simple Function

(29) Egorov's Theorem and Lusin's Theorem

(30) The Lebesgue Integral. This is done in 4 stages: Simple Functions, Bounded Functions with finite support, Non-Negative Functions, and General Functions.

(31) The counterexample to $\int f_n \nrightarrow \int f$ from Lecture 21

(32) The Bounded Convergence Theorem

(33) Fatou's Lemma

(34) $f_n \nearrow f$ and the Monotone Convergence Theorem

(35) Interchange of limits and series

(36) The Dominated Convergence Theorem

(37) L^1, $\|f\|_{L_1}$, and more generally L^p and $\|f\|_{L_p}$ for $1 \le p < \infty$

Trig Formulas you should know

Here is a list of trig identities you should know. Anything else will be provided

(1) $\cos^2(x) + \sin^2(x) = 1$

(2) $1 + \tan^2(x) = \sec^2(x)$

(3) Derivatives of $\sin, \cos, \tan, \sec$

(4) Antiderivatives of $\sin, \cos, \tan$

(5) $\cos(-x) = \cos(x), \sin(-x) = -\sin(x)$, $\tan(-x) = -\tan(x)$

(6) $\cos(2x) = \cos^2(x) - \sin^2(x)$

(7) $\sin(2x) = 2\sin(x)\cos(x)$

(8) $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

(9) $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

(10) $\int \cos^2(x)dx, \int \sin^2(x)dx$

List of formulas that will be provided

Here is a list of formulas that I will provide if necessary. You do not need to memorize them.

(1) Fourier series and coefficients for intervals *other* than $[-\pi, \pi]$

(2) D_N and its explicit formula

(3) The Fourier Transform and all its properties, but I will not give you Self-Adjointness or the properties of $f \star g$

(4) The definition of a good kernel

(5) The definition of the Schwarz space

(6) All the formulas for $\cos(A)\cos(B)$, $\sin(A)\sin(B)$, $\cos(A)\sin(B)$

(7) $\|f\|_{L^\infty}$

Things you can ignore

- Lecture 1: The proof in the section "Uniform Convergence and Integration"
- Lecture 1: The $nx\left(1-x^2\right)^n$ example **but** I could give you the formula and ask you to show $f_n \to 0$ pointwise on $[0,1]$ but $\int_0 f_n \not\to 0$
- Lecture 2: The proof in the section "Uniform Convergence and Differentiation"
- Lecture 2: The $f_n = \frac{x^2}{(1+x^2)^n}$ example **but** I could ask you to show what $\sum f_n$ converges to.

- Lecture 3: The "Counterexample" section showing that $\sin(nx)$ has no convergent subsequence
- Lecture 3: The proof of the Arzelà-Ascoli Theorem
- Lecture 4: The proof of the Picard-Lindelöf Theorem, but notice how we wrote the ODE as a fixed point problem
- Lecture 4: For the continuous but nowhere differentiable function, I just want you to understand how the Weierstraß M test is used, the rest you can safely ignore
- Lecture 5: The proof of the Weierstraß Approximation Theorem
- Lecture 5: The terminology of the Stone-Weierstraß Theorem, I would give it to you if necessary
- Lecture 5: The proof of term-by-term integration and differentiation of series
- Lecture 6: The proof of Fubini for series
- Lecture 6: The statement and proof of Taylor's theorem
- Lecture 6: The section on the Gamma Function and the section on the Half-Derivative
- Lecture 7: The proof of uniqueness of Fourier series
- Lecture 8: The proof of the formula for D_N
- Lecture 8: The proof of pointwise convergence (the one with Lipschitz at x)
- Lecture 9: The proof of the Best Approximation Lemma and the proof of Mean-Squared convergence

- Lecture 10: The proofs in the section "Derivatives and Fourier transforms"
- Lecture 10: The proof of the Fourier Inversion Formula and of Plancherel's Theorem
- Lecture 11: You don't need to memorize the condition with $\|B - A\| \left\|A^{-1}\right\| < 1$, and you can skip the proof, but just know that "if A is invertible and B is close to A, then B is invertible."
- Lecture 11: The proof that $A \to A^{-1}$ is continuous
- Lecture 12: The proof that derivatives are unique
- Lecture 12: The proof that if the partial derivatives exist and are continuous, then f is differentiable
- Lecture 13: The part of the proof of the Mean Value Theorem starting from "Claim."
- Lecture 15: The proof of the Inverse Function Theorem, but just notice that it's an application of the Banach Fixed Point Theorem
- Lecture 16: The proof of the Implicit Function Theorem **except** for **STEPS 1, 2** (up to "f invertible"), and just the differentiation formula in **STEP 5**.
- Lecture 16: The Rank Theorem
- Lecture 17: The Rectangle Lemma and the proof of Clairaut's theorem
- Lecture 17: The true second derivative test

- Lecture 17: For differentiation of integrals, all I want you to notice is how you're using a difference quotient, and uniform convergence
- Lecture 17: An interesting example
- Lecture 18: The first fact about disjoint unions of rectangles
- Lecture 18: The proof of Property 5 (I skipped that in lecture)
- Lecture 19: The statement and proof with $m(E\Delta F) \leq \epsilon$ (I skipped that in lecture) and the proof of Proposition 4 in the Appendix
- Lecture 20: The Approximation Theorem with the truncation, but the one with $|\phi_k|$ you should know
- Lecture 20: Proof of Independence of Representation in the Lebesgue Integral
- Lecture 21: The comparison between Riemann and Lebesgue integrals, but all I really want you to notice is the "Upshot" part of that proof, and how you can use the BCT to pass through the limit
- Lecture 22: You don't need to memorize the statements of the section "Regularity," but you need to know their proofs
- Lecture 22: The proof that L^1 is complete
- Homework 4: Ignore Additional Problems 1 and 2
- Homework 5: Ignore Additional Problems 2, 3, and 5
- Homework 6: Ignore Problem 14

- Homework 7: Ignore Problem 16 and Additional Problem 1
- Homework 8: Ignore Problem 28
- Homework 9: Ignore Additional Problems 2 and 4