MATH S4062 – FINAL EXAM

Name	
UNI	
Signature	

Instructions: This is it, your final hurdle to freedom! You have 95 minutes to take this exam, for a total of 50 points. No books, notes, calculators, or cellphones are allowed. **Please write in complete sentences if you can.** Remember that you are not only graded on your final answer, but also on your work. If you need to continue your work on the back of the page, please clearly indicate so.

Academic Honesty Statement: With the signature above, I certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating results in an automatic F in the course, and will be further subject to disciplinary consequences, pursuant to the Columbia University Honor Code.

Date: Friday, August 12, 2022.

1. *(10 points)* Prove the Bounded Convergence Theorem:

If $\{f_n\}$ is a sequence of measurable functions bounded by M and supported on a set E of finite measure with $f_n \to f$ a.e.

$$\text{Then } \lim_{n\to\infty} \int |f_n - f|\, dx = 0$$

2. *(10 points)* Prove Markov's Theorem:

If $f \in L^p\left(\mathbb{R}^d\right)$ with $1 \leq p < \infty$ then for every $t > 0$ we have

$$m\{x \mid |f(x)| > t\} \leq \frac{1}{t^p}\left(\int |f|^p \, dx\right)$$

3. *(10 points)* Consider the system

$$\begin{cases} x^2 - y^2 + 3z - u^3 + v^2 = -4 \\ 2xy + y^2 - 4z - 2u^2 + 3v^4 = -8 \end{cases}$$

Show that you can solve for u and v in terms of x, y, z around the point $(2, -1, 0, 2, 1)$ and calculate $G'(2, -1, 0)$, where G is the graph of u, v in terms of x, y, z

4. *(10 points)* Let $\{K_n\}_{n=1}^{\infty}$ be a family of functions called **good kernels** with the following properties:

   (1) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1$

   (2) There is $M > 0$ such that for all n, $\int_{-\pi}^{\pi} |K_n(x)|\, dx \leq M$

   (3) For every $\delta > 0$, $\lim_{n\to\infty} \int_{\delta\leq|x|\leq\pi} |K_n(x)|\, dx = 0$

   Let f be a 2π periodic function that is continuous at x, show

$$\lim_{n\to\infty} (f \star K_n)(x) = f(x)$$

5. *(10 points)* Show that the (following version of the) Weierstraß approximation theorem is **false** for $\mathbb{R}$:

Weierstraß Approximation Theorem: If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then for all $\epsilon > 0$ there is a polynomial p such that $|f(x) - p(x)| < \epsilon$ for all x