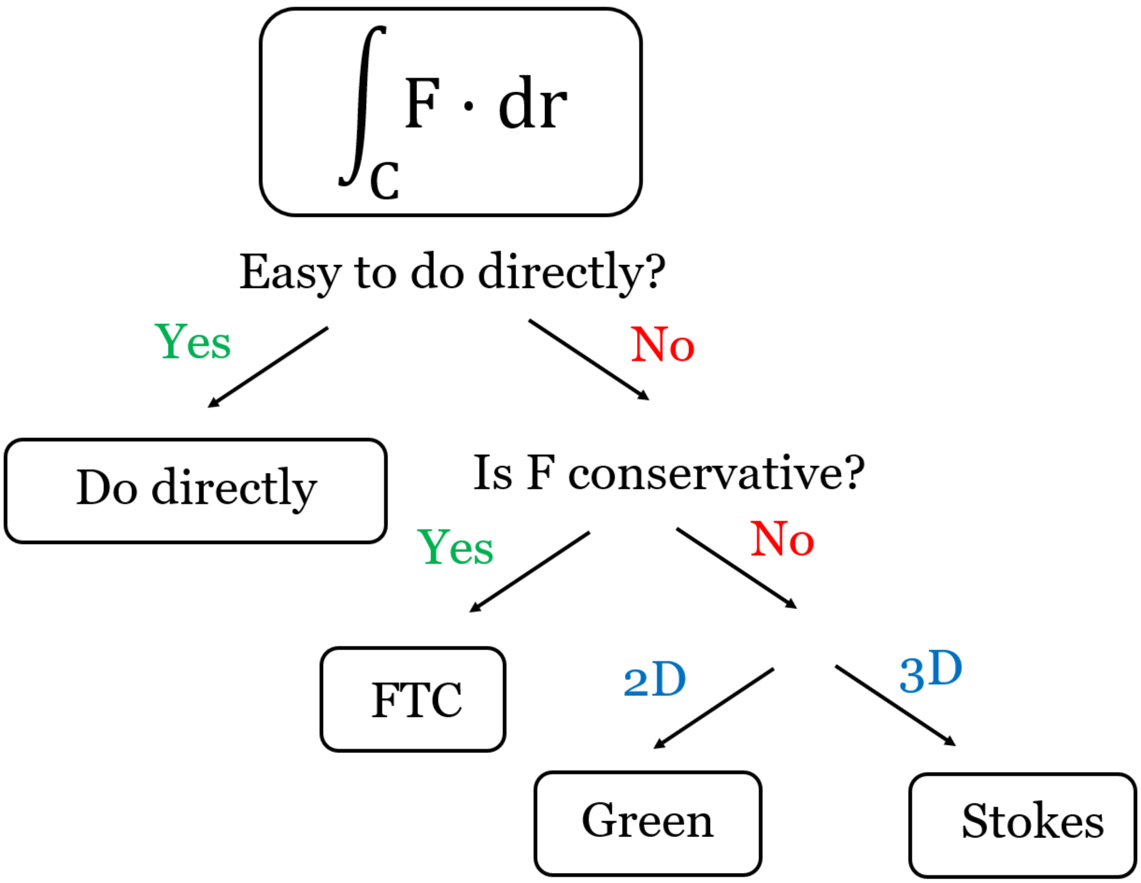
FINAL EXAM REVIEW: LINE INTEGRALS

Welcome to the first part of our final exam review session! Today is all about what to do if you have to calculate a line integral. Fortunately there is a nice roadmap for what theorem to use when:



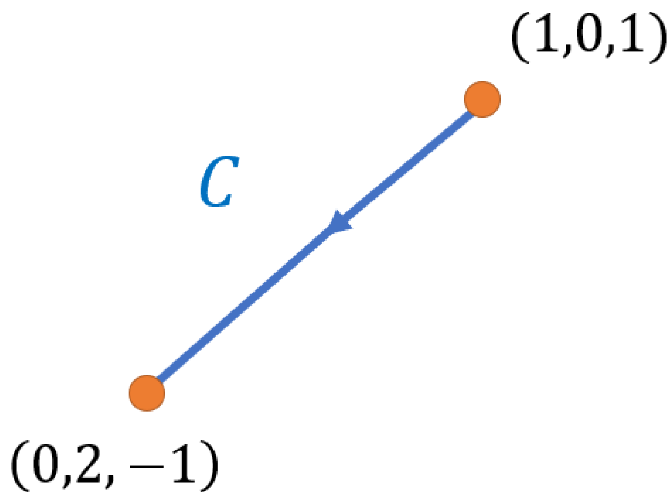
Date: Thursday, December 9, 2021.

1. Do Directly

Example 1:

Calculate $\int_C F \cdot dr$, where $F = \left\langle x^2, 3y, 3xz \right\rangle$, and C is the line from $(1,0,1)$ to $(0,2,-1)$

STEP 1: Picture:



STEP 2: Parametrize C:

$$\begin{cases} x(t) = & (1-t)1 + t(0) = 1-t \\ y(t) = & (1-t)0 + t(2) = 2t \\ z(t) = & (1-t)1 + t(-1) = 1-2t \end{cases}$$

So $r(t) = \langle 1-t, 2t, 1-2t \rangle$ with $0 \leq t \leq 1$

STEP 3:

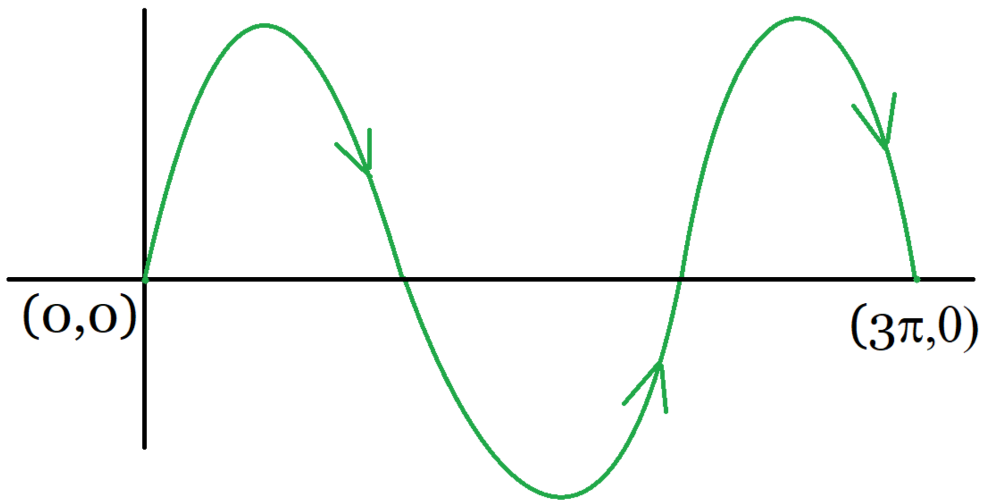
$$\begin{aligned}
\int_C F \cdot dr &= \int_0^1 F(r(t)) \cdot r'(t)dt \\
&= \int_0^1 \underbrace{\left\langle (1-t)^2, 3(2t), 3(1-t)(1-2t) \right\rangle}_{\langle x^2, 3y, 3xz \rangle} \cdot \langle -1, 2, -2 \rangle \, dt \\
&= \int_0^1 -(1-t)^2 + 12t - 6(1-t)(1-2t)dt \\
&= \int_0^1 \left(-1 + 2t - t^2\right) + 12t + \left(-6 + 12t + 6t - 12t^2\right) dt \\
&= \int_0^1 -13t^2 + 32t - 7dt \\
&= \left[-\frac{13}{3}t^3 + 16t^2 - 7t \right]_0^1 \\
&= -\frac{13}{3} + 16 - 7 \\
&= -\frac{13}{3} + 9 \\
&= \frac{14}{3}
\end{aligned}$$

2. FTC for Line Integrals

Example 2:

Evaluate $\int_C F \cdot dr$, where $F = \left\langle xy^2, x^2y \right\rangle$, and C is the arc of $y = \sin(x)$ from $(0,0)$ to $(3\pi, 0)$

STEP 1:



STEP 2: F **conservative?**

You have to check for this anyway, so it's **NOT** a waste of time.

$$\text{Check: } Q_x - P_y = \left(x^2y\right)_x - \left(xy^2\right)_y = 2xy - 2xy = 0\checkmark$$

STEP 3: Find f

$$F = \nabla f \Rightarrow \left\langle xy^2, x^2y \right\rangle = \langle f_x, f_y \rangle$$

$$f_x = xy^2 \Rightarrow f(x,y) = \frac{1}{2}x^2y^2 + \text{ JUNK}$$
$$f_y = x^2y \Rightarrow f(x,y) = \frac{1}{2}x^2y^2 + \text{ JUNK}$$

Hence $f(x,y) = \frac{1}{2}x^2y^2$

STEP 4: By the FTC:

$$\int_C F \cdot dr = f(3\pi, 0) - f(0,0) = \frac{1}{2}(3\pi)^2 0^2 - \frac{1}{2}0^2 0^2 = 0$$

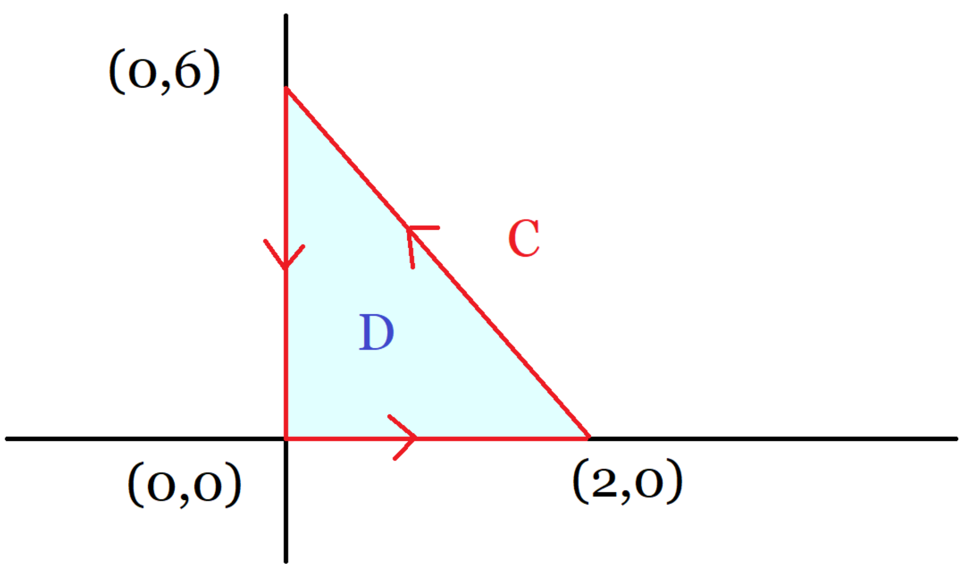
Note: In 3 dimensions, you need to check $\text{curl}(F) = \langle 0,0,0 \rangle$ (just like in the lecture on curl or the practice exams)

3. Green's Theorem

Example 3:

Evaluate $\int_C F \cdot dr$, where $F = \left\langle x^3, 2xy \right\rangle$ and C is the triangle with vertices $(0,0), (2,0), (0,6)$ (counterclockwise)

STEP 1: Picture



(Too painful to do it directly)

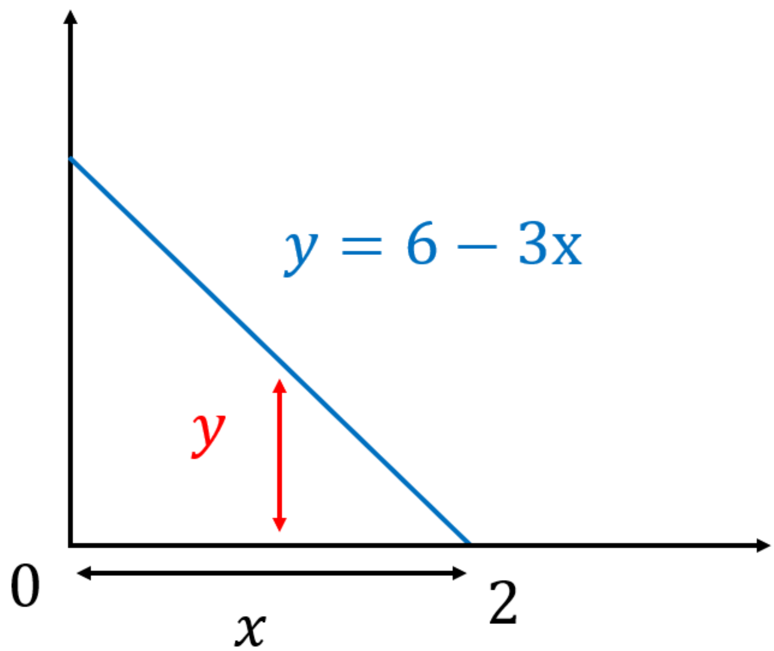
STEP 2: F **conservative?**

$$Q_x - P_y = (2xy)_x - \left(x^3\right)_y = 2y$$

NOT Conservative (and 2 dimensions) $\Rightarrow$ Green!

STEP 3: By Green's Theorem:

$$\begin{aligned}
\int_C F \cdot dr &= \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy \\
&= \int\int_D 2y - xdxdy \\
&= \int_0^2 \int_0^{6-3x} 2ydydx \\
&= \int_0^2 \left[y^2\right]_{y=0}^{y=6-3x} dx \\
&= \int_0^2 (6-3x)^2 \, dx \\
&= \left[\frac{1}{3}\left(\frac{1}{-3}\right)(6-3x)^3\right]_0^2 \\
&= -\frac{1}{9}(6-3(2))^3 + \frac{1}{9}6^3 \\
&= \frac{6 \times 6 \times 6}{9} \\
&= 2 \times 2 \times 6 \\
&= 24
\end{aligned}$$



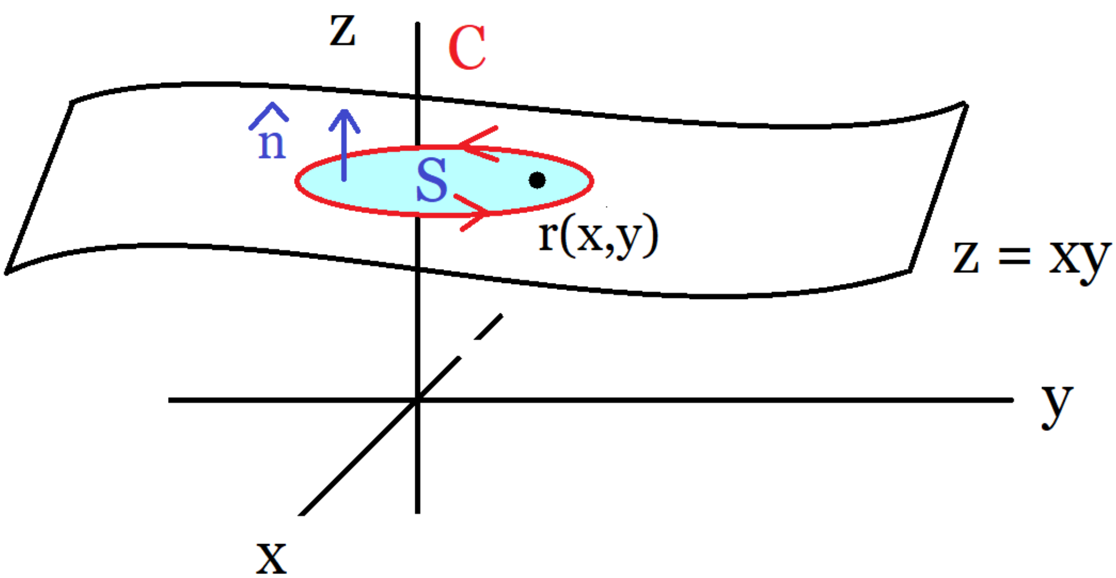
4. Stokes' Theorem

Example 4:

Evaluate $\int_C F \cdot dr$, where $F = \langle \sin(x), \cos(y), xz \rangle$ and C Curve parametrized by $r(t) = \langle \cos(t), \sin(t), \cos(t)\sin(t) \rangle$ with $0 \leq t \leq 2\pi$

Hint: C lies on the surface $z = xy$

STEP 1: Picture



STEP 2: F **conservative?**

$$\begin{aligned}
\operatorname{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(x) & \cos(y) & xz \end{vmatrix} \\
&= \left\langle \frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(\cos(y)), -\frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial z}(\sin(x)) \right. \\
&\quad \left. , \frac{\partial}{\partial x}(\cos(y)) - \frac{\partial}{\partial y}(\sin(x)) \right\rangle \\
&= \langle 0, -z, 0 \rangle \\
&\neq \langle 0, 0, 0 \rangle
\end{aligned}$$

STEP 3: Stokes' Theorem

$$\int_C F \cdot dr = \int \int_S \operatorname{curl}(F) \cdot d\mathbf{S}$$

STEP 4: Parametrize S**:**

$$r(x, y) = \langle x, y, xy \rangle$$

$$r_x = \langle 1, 0, y \rangle$$
$$r_y = \langle 0, 1, x \rangle$$

$$\hat{n} = r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = \left\langle -y, -x, \underbrace{1}_{\geq 0} \right\rangle \checkmark$$

STEP 5: By Stokes:

$$\begin{aligned}
\int\int_S \operatorname{curl}(F) \cdot d\mathbf{S} &= \int\int_D \underbrace{\langle 0, -xy, 0 \rangle}_{\langle 0, -z, 0 \rangle} \cdot \underbrace{\langle -y, -x, 1 \rangle}_{\hat{n}} dxdy \\
&= \int\int_D x^2 y dxdy \qquad \text{D: Disk of Radius 1} \\
&= \int_0^{2\pi} \int_0^1 r^2 \cos^2(\theta) r \sin(\theta) r dr d\theta \\
&= \left(\int_0^1 r^4 dr \right) \left(\int_0^{2\pi} \cos^2(\theta) \sin(\theta) d\theta \right) \\
&= \frac{1}{5} \left[-\frac{\cos^3(\theta)}{3} \right]_0^{2\pi} \\
&= 0
\end{aligned}$$