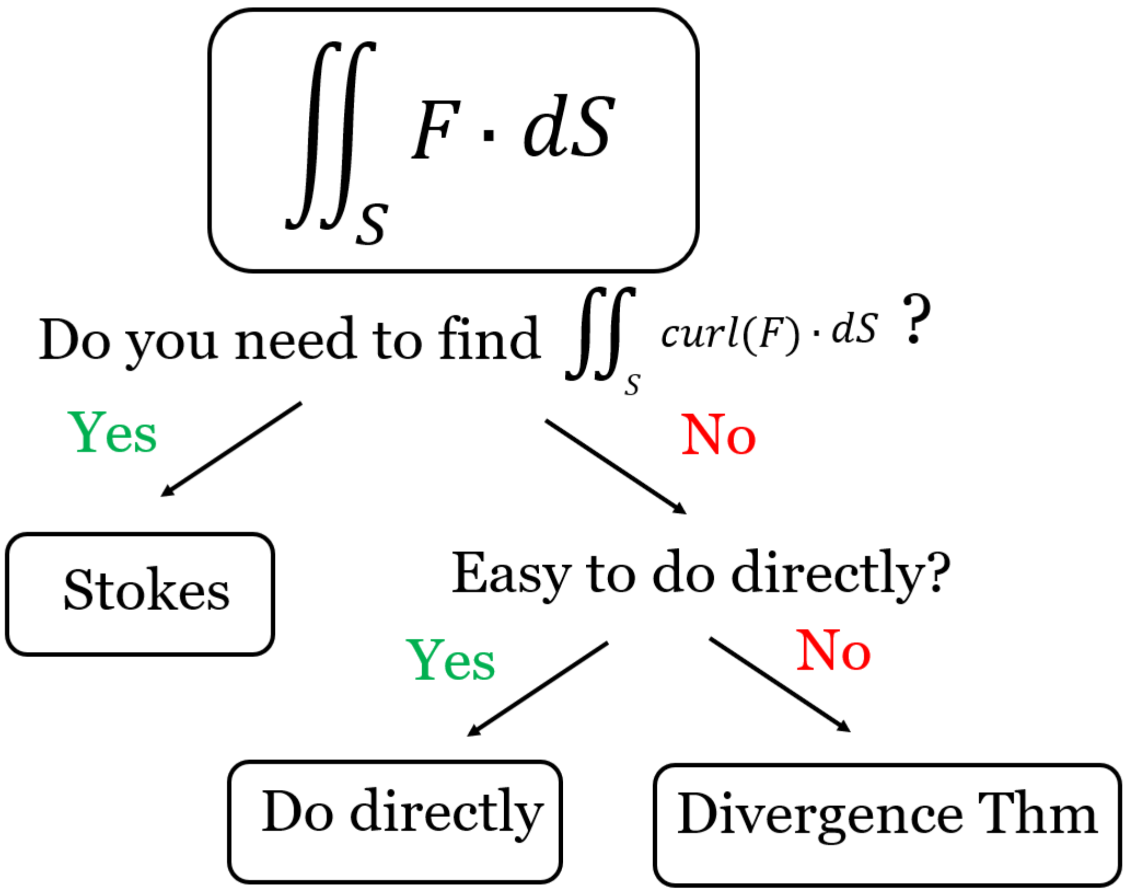
FINAL EXAM REVIEW: SURFACE INTEGRALS

Welcome to the second part of our final exam review session! Today is all about what to do if you have to calculate a surface integral. Fortunately there is a another roadmap for this



Date: Thursday, December 9, 2021.

1. Stokes' Theorem

Example 5:

(a) Find F such that $G = \text{curl}(F)$, where $G = \langle 0, y, -z\rangle$

Hint: Guess $F = \langle P, 0, 0\rangle$ for some P

$$\begin{aligned}
\text{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & 0 & 0 \end{vmatrix} \\
&= \left\langle \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(0), -\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial z}P, \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(P) \right\rangle \\
&= \langle 0, P_z, -P_y\rangle \\
&\overset{\text{WANT}}{=} \langle 0, y, -z\rangle
\end{aligned}$$

Therefore:

$$\begin{cases} P_z = y \Rightarrow P = \int y dz = yz + \text{ JUNK} \\ P_y = z \Rightarrow P = \int z dy = zy + \text{ JUNK} \end{cases}$$

Hence $P = zy$, so

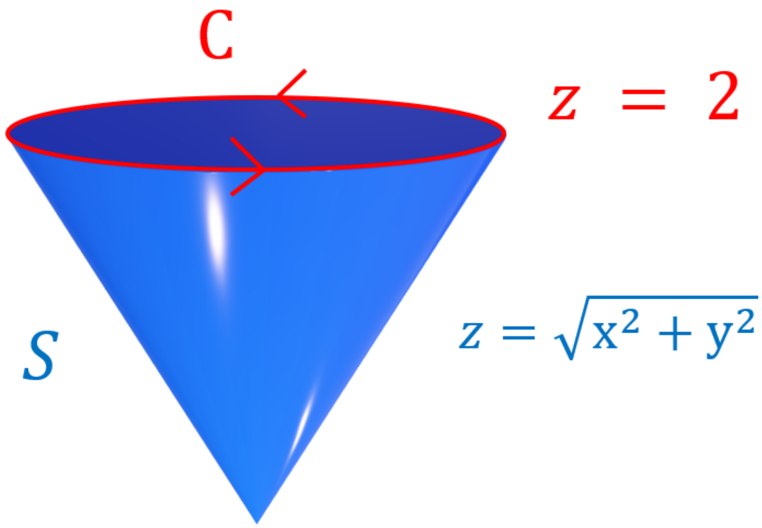
$$F = \langle P, 0, 0\rangle = \langle yz, 0, 0\rangle$$

(b) $\int\int_S G \cdot d\mathbf{S}$ (G as in (a))

S is the part of the surface $z = \sqrt{x^2 + y^2}$ and $0 \leq z < 2$ (assume counterclockwise orientation)

STEP 1: Picture:

Notice that $z = \sqrt{x^2+y^2} \Rightarrow z^2 = x^2 + y^2$ (Cone)



STEP 2:

$$\int\int_S G \cdot d\mathbf{S} = \int\int_S \text{curl}(F) \cdot d\mathbf{S} \text{ (By (a))} = \int_C F \cdot dr \quad \text{(Stokes)}$$

Where $F = \langle yz, 0, 0 \rangle$

STEP 3: What is C?

$$z = \sqrt{x^2+y^2} \text{ and } z = 2 \Rightarrow \sqrt{x^2+y^2} = 2 \Rightarrow x^2 + y^2 = 4$$

So C is a circle of radius 2, in the clockwise direction

$$r(t) = \langle 2\cos(t), 2\sin(t), 2 \rangle, \quad (0 \le t \le 2\pi)$$

STEP 4:

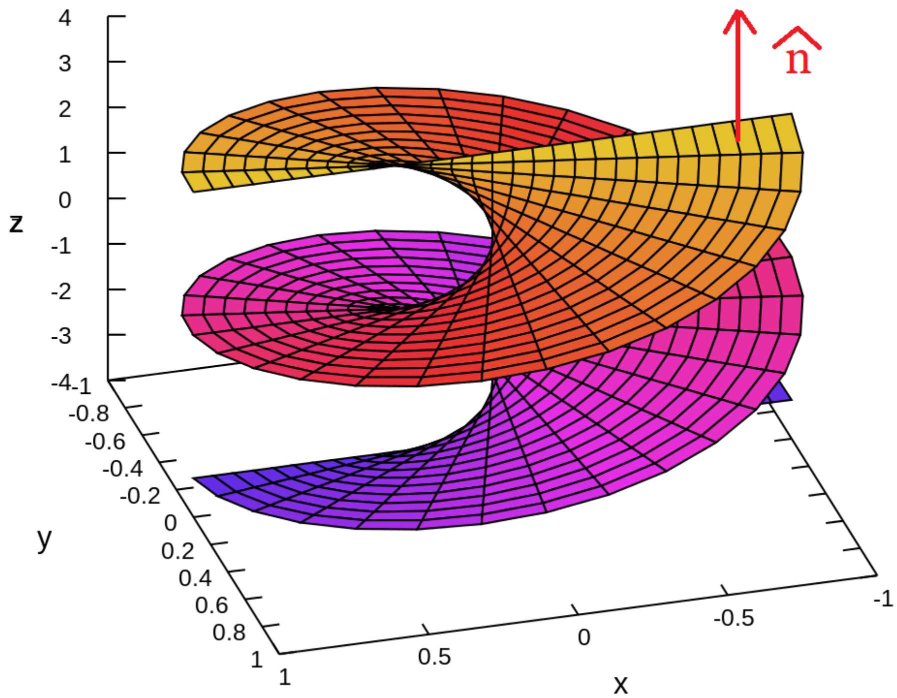
$$
\begin{aligned}
\int_C F \cdot dr &= \int_0^{2\pi} F(r(t)) \cdot r'(t)dt \\
&= \int_0^{2\pi} \underbrace{\langle 2\sin(t)(2), 0, 0\rangle}_{\langle yz,0,0\rangle} \cdot \underbrace{\langle -2\sin(t), 2\cos(t), 0\rangle}_{r'(t)} dt \\
&= \int_0^{2\pi} -8\sin^2(t)dt \\
&= -8\int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos(2t)dt \\
&= -8\left[\frac{t}{2} - \frac{1}{4}\sin(2t)\right]_0^{2\pi} \\
&= -8\left(\pi - 0 - \frac{1}{4}\sin(4\pi) + \frac{1}{4}\sin(0)\right) \\
&= -8\pi
\end{aligned}
$$

2. Do Directly

Example 6:

Calculate $\int\int_S F \cdot d\mathbf{S}$ where $F = \left\langle x, y, z^2 \right\rangle$ and S is the Helicoid parametrized by $r(u,v) = \langle u\cos(v), u\sin(v), v\rangle$ with $0 \leq u \leq 2$ and $0 \leq v \leq 4\pi$

STEP 1: Picture:



STEP 2: Parametrize S:

$$r(u, v) = \langle u\cos(v), u\sin(v), v\rangle$$

STEP 3: Normal Vector:

$$r_u = \langle \cos(v), \sin(v), 0\rangle$$
$$r_v = \langle -u\sin(v), u\cos(v), 1\rangle$$

$$
\begin{aligned}
\hat{n} =& r_u \times r_v \\
=& \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 1 \end{vmatrix} \\
=& \left\langle \sin(v), -\cos(v), u\cos^2(v) + u\sin^2(v) \right\rangle \\
=& \left\langle \sin(v), -\cos(v), \underbrace{u}_{\geq 0} \right\rangle
\end{aligned}
$$

STEP 4:

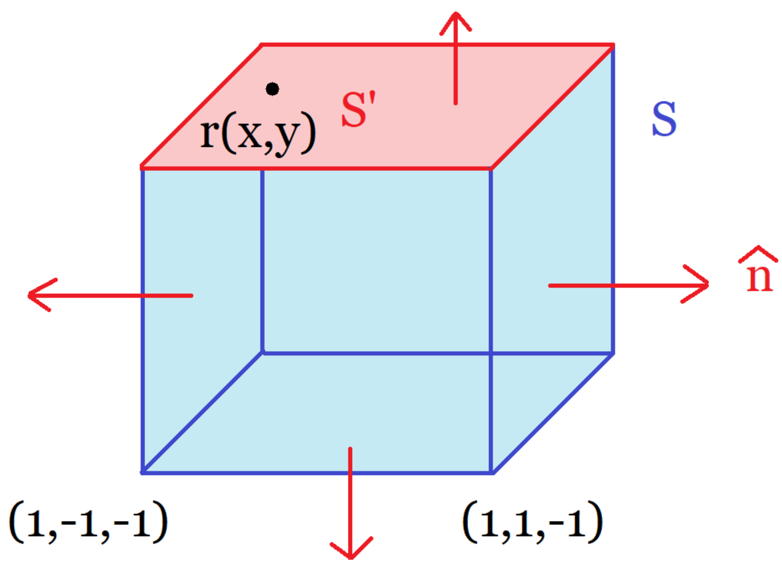
$$
\begin{aligned}
\int\int_S F \cdot d\mathbf{S} =& \int\int_D F \cdot \hat{n}\, dudv \\
=& \int\int_D \underbrace{\left\langle u\cos(v), u\sin(v), v^2 \right\rangle}_{\langle x,y,z^2\rangle} \cdot \underbrace{\langle \sin(v), -\cos(v), u\rangle}_{\hat{n}} dudv \\
=& \int_0^{4\pi}\int_0^2 \cancel{u\cos(v)\sin(v)} - \cancel{u\sin(v)\cos(v)} + uv^2 dudv \\
=& \left(\int_0^2 udu\right)\left(\int_0^{4\pi} v^2 dv\right) \\
=& \frac{2^2}{2}\left(\frac{(4\pi)^3}{3}\right) \\
=& \frac{128\pi^3}{3}
\end{aligned}
$$

3. Divergence Theorem

Example 7: (Russian, 7 = Siem)

Evaluate $\int\int_S F \cdot d\mathbf{S}$, where $F = \langle xy^2, x^2z, e^y \rangle$ where S is the Cube with vertices $(\pm 1, \pm 1, \pm 1)$ oriented outwards, without the top

STEP 1: Picture:



STEP 2:

Warning

The Divergence Theorem only holds for closed surfaces!

Let S' = top of cube, then $S + S'$ is closed, so by the divergence theorem

$$\begin{aligned}
\int\int_{S+S'} F \cdot d\mathbf{S} =& \int\int\int_E \text{div}(F) dxdydz \\
=& \int\int\int_E \left(xy^2\right)_x + \left(x^2 z\right)_y + \left(e^y\right)_z dxdydz \\
=& \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} y^2 dxdydz \\
=& (2)(2)\left[\frac{y^3}{3}\right]_{-1}^{1} \\
=& 4\left(\frac{1}{3} - \frac{(-1)^3}{3}\right) \\
=& \frac{8}{3}
\end{aligned}$$

STEP 3: $\int\int_{S'} F d\mathbf{S}$

Parametrize S'**:** $r(x,y) = \langle x, y, 1\rangle$

$$\begin{aligned}
r_x &= \langle 1, 0, 0\rangle \\
r_y &= \langle 0, 1, 0\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{n} =& r_x \times r_y \\
=& \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\
=& \left\langle 0, 0, \underbrace{1}_{\geq 0} \right\rangle
\end{aligned}$$

$$\begin{aligned}\int\int_{S'} F \cdot d\mathbf{S} =& \int\int_D \underbrace{\left\langle xy, x^2(1), e^y \right\rangle}_{\langle xy^2, x^2 z, e^y\rangle} \cdot \underbrace{\langle 0,0,1\rangle}_{\hat{n}} dxdy \\ =& \int_{-1}^{1}\int_{-1}^{1} e^y dxdy \\ =& 2\int_{-1}^{1} e^y dy \\ =& 2\left(e - e^{-1}\right)\end{aligned}$$

STEP 4: Answer:

$$\int\int_S F \cdot d\mathbf{S} = \int\int_{S+S'} F \cdot d\mathbf{S} - \int\int_{S'} F \cdot d\mathbf{S} = \frac{8}{3} - 2\left(e - e^{-1}\right)$$