SOLUTIONS

**MATH 2D – FINAL**

Name: ______________________

Student ID: ______________________

Lecture time: (please circle) 2 – 3 PM 4 – 5 PM

**Instructions:** This is it, your final hurdle to freedom!!! You have 120 minutes to take this exam, for a total of 100 points. This is a closed book and closed notes exam and calculators and/or portable electronic devices are not allowed. Remember that you are not only graded on your final answer, but also on your work. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. May the Chen Lou be with you!!!

**Academic Honesty Statement:** I hereby certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating will be subject to disciplinary consequences, pursuant to section 102.1 of the UCI Student Code of Conduct.

**Signature:** ______________________________

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|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| Total | | 100 |

*Date*: Wednesday, June 13, 2018.

1. *(10 points)* Find parametric equations of the tangent line to the following curve at $(0, 1, -1)$, where $0 \le t \le 2\pi$. Simplify your answer.

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), \cos(2t) \rangle$$

1) FIND $t$

$(\cos(t), \sin(t), \cos(2t)) = (0, 1, -1)$

$$\Rightarrow \begin{cases} \cos(t) = 0 \\ \sin(t) = 1 \\ \cos(2t) = -1 \end{cases} \quad \text{AND} \quad 0 \le t \le 2\pi$$

$t = \frac{\pi}{2}$ WORKS

2) DIRECTION VECTOR

$r'(t) = \langle -\sin(t), \cos(t), -2\sin(2t) \rangle$

$r'\left(\frac{\pi}{2}\right) = \langle -\sin\left(\frac{\pi}{2}\right), \cos\left(\frac{\pi}{2}\right), -2\sin(\pi) \rangle$

$= \langle -1, 0, 0 \rangle$

3) POINT $(0, 1, -1)$

4) EQUATIONS :

$$\begin{cases} x(t) = 0 + (-1)t \\ y(t) = 1 + 0t \\ z(t) = -1 + 0t \end{cases} \Rightarrow \boxed{\begin{cases} x(t) = -t \\ y(t) = 1 \\ z(t) = -1 \end{cases}}$$

2. *(10 points)* Find the (smallest) angle between the following two planes

$$x + y = 2 \text{ and } y - z = 1$$

1) NORMAL VECTORS

$N_1 = \langle 1, 1, 0 \rangle$ (BECAUSE $1x + 1y + 0z = 2$)

$N_2 = \langle 0, 1, -1 \rangle$ (BECAUSE $0x + 1y + (-1)z = 1$)

2) $N_1 \cdot N_2 = \|N_1\| \|N_2\| \cos(\theta)$

$N_1 \cdot N_2 = (1)(0) + (1)(1) + (0)(-1) = 1$

$\|N_1\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$

$\|N_2\| = \sqrt{0^2 + 1^2 + (-1)^2} = \sqrt{2}$

3) HENCE $1 = (\sqrt{2})(\sqrt{2}) \cos(\theta)$

$\cos(\theta) = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

3. *(10 points)* Find $g_{rs}$, where

$$g = g(r+s, r-s)$$

**Note:** Simplify your answer as much as possible.

$$g_r = g_x\,(r+s)_r + g_y\,(r-s)_r$$
$$= g_x(r+s, r-s) + g_y(r+s, r-s)$$

$$g_{rs} = (g_r)_s$$
$$= \left(g_x(r+s, r-s)\right)_s + \left(g_y(r+s, r-s)\right)_s$$
$$= g_{xx}\,(r+s)_s + g_{xy}\,(r-s)_s + g_{yx}\,(r+s)_s + g_{yy}\,(r-s)_s$$
$$= g_{xx} - \cancel{g_{xy}} + \cancel{g_{yx}} - g_{yy}$$
$$= \boxed{g_{xx} - g_{yy}}$$

4. *(10 points)* Assume that $c > 0$ is a fixed constant. Show that the sum of the $x, y,$ and $z-$intercepts (assuming they exist) to any tangent plane to the following surface is equal to $c$:

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$$

**Note:** You only need show your work for one of the intercepts.

1) $F(x,y,z) = \sqrt{x} + \sqrt{y} + \sqrt{z} - \sqrt{c}$

2) EQUATION $F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$

$F_x = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x_0}}$ AT $(x_0,y_0,z_0)$ AND SIMILARLY $F_y = \frac{1}{2\sqrt{y_0}}$, $F_z = \frac{1}{2\sqrt{z_0}}$

HENCE WE GET $\frac{1}{\cancel{2}\sqrt{x_0}}(x-x_0) + \frac{1}{\cancel{2}\sqrt{y_0}}(y-y_0) + \frac{1}{\cancel{2}\sqrt{z_0}}(z-z_0) = 0$ ⌋ 4

3) X-INTERCEPT (SET $y=0$, $z=0$)

$\frac{1}{\sqrt{x_0}}(x-x_0) + \frac{1}{\sqrt{y_0}}(-y_0) + \frac{1}{\sqrt{z_0}}(-z_0) = 0$

$\frac{x}{\sqrt{x_0}} - \frac{x_0}{\sqrt{x_0}} - \sqrt{y_0} - \sqrt{z_0} = 0$

$\frac{x}{\sqrt{x_0}} = \sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} \Rightarrow x = \sqrt{x_0}\,(\overbrace{\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}}^{\sqrt{c} \text{ SINCE } (x_0,y_0,z_0) \text{ IS ON THE SURFACE}})$

$\Rightarrow x = \sqrt{x_0}\,\sqrt{c}$

SIMILARLY, Y-INT $y = \sqrt{y_0}\sqrt{c}$

Z-INT $z = \sqrt{z_0}\sqrt{c}$ ⌋ 4

4) SUM $x + y + z = \sqrt{x_0}\sqrt{c} + \sqrt{y_0}\sqrt{c} + \sqrt{z_0}\sqrt{c}$

$= \sqrt{c}\,(\underbrace{\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}}_{\sqrt{c}})$

$= (\sqrt{c})(\sqrt{c})$

$= c$ ✓ ⌋ 2

5. *(10 points)*

(a) *(5 points)*

Classify the critical point(s) of the function

$$f(x, y) = x^2 + 4y^2 - 6x$$

1) $f_x = 2x - 6 = 0 \Rightarrow x = 3$

$f_y = 8y = 0 \Rightarrow y = 0$

CP $(3,0)$ 3

2) $D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 8 \end{vmatrix}$

$D(3,0) = \begin{vmatrix} 2 & 0 \\ 0 & 8 \end{vmatrix} = 16 > 0$ AND $f_{xx}(3,0) = 2 > 0$

SO $f$ HAS A LOCAL MIN AT $(3,0)$

2

(b) *(5 points)* Find the absolute maximum and minimum values of the function $f$ in $(a)$ ~~on~~ IN the disk $x^2 + y^2 \leq 1$.

1) ⚠ THE CP $(3,0)$ IN $(a)$ IS NOT IN THE DISK $x^2+y^2 \leq 1$, SO IGNORE IT!

2) LAGRANGE $f(x,y) = x^2 + 4y^2 - 6x$, $g(x,y) = x^2+y^2-1$

$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases} \Rightarrow \begin{cases} 2x - 6 = 2\lambda x \\ 8y = 2\lambda y \end{cases} \Rightarrow \begin{cases} x - 3 = \lambda x \\ 4y = \lambda y \end{cases} \Rightarrow \begin{cases} x(1-\lambda) = 3 \\ y(4-\lambda) = 0 \end{cases}$ 2

$\Rightarrow \begin{cases} x = \dfrac{3}{1-\lambda} \\ y = 0 \text{ OR } \lambda = 4 \end{cases}$

CASE 1 $y = 0$, THEN BY $x^2+y^2=1$ GET $x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow (1,0)$ & $(-1,0)$

CASE 2 $\lambda = 4$, THEN $x = \dfrac{3}{1-\lambda} = -1$ AND FROM $x^2+y^2=1$ GET $y = 0 \Rightarrow (-1,0)$

3) COMPARE $f(1,0) = 1 - 6 = -5 \rightarrow$ ABS MIN

$f(-1,0) = 1 + 6 = 7 \rightarrow$ ABS MAX

3

6. *(10 points)*

(a) *(8 points)* Use **Lagrange multipliers** to find the absolute maximum of the following function subject to the following constraint, where $x, y, z > 0$ and $c > 0$ is a fixed constant.

$f(x, y, z) = xyz$ subject to $x + y + z = c$ $\Rightarrow g(x,y,z) = x+y+z-c$

1) LAGRANGE

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \end{cases} \Rightarrow \begin{cases} yz = \lambda \\ xz = \lambda \\ xy = \lambda \end{cases}$$

①, ② 2

① $y\cancel{z} = x\cancel{z} \Rightarrow y = x$

② $\cancel{x}z = \cancel{x}y \Rightarrow z = y$

$x = y = z$ 2

2) CONSTRAINT $x + y + z = c \Rightarrow 3x = c \Rightarrow x = y = z = \frac{c}{3}$ 2

3) ABS MAX $f\left(\frac{c}{3}, \frac{c}{3}, \frac{c}{3}\right) = \left(\frac{c}{3}\right)^3 = \frac{c^3}{27}$ 2

(b) *(2 points)* Use (a) to show that for all $x, y, z > 0$

$$(xyz)^{\frac{1}{3}} \leq \frac{x + y + z}{3}$$

SINCE $\left(\frac{c}{3}\right)^3$ IS THE ABS MAX OF $f(x,y,z) = xyz$,

WE GET $xyz \leq \left(\frac{c}{3}\right)^3$ 1

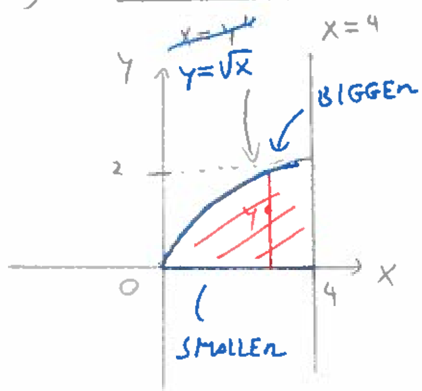
$\Rightarrow (xyz)^{\frac{1}{3}} \leq \frac{c}{3}$ SINCE $c = x+y+z$

$\Rightarrow \boxed{(xyz)^{\frac{1}{3}} \leq \frac{x+y+z}{3}}$ 1

7. *(10 points)* Calculate

$$\int_0^2 \int_{y^2}^{4} y\cos(x^2)\,dx\,dy$$

1) CHANGE THE ORDER




KNOW $\quad y^2 \le x \le 4$

$0 \le y \le 2$

$(x = y^2 \Rightarrow y = \sqrt{x})$

WANT $\quad ? \le y \le ?$

$? \le x \le ?$ (CONSTANTS)

SMALLER $\le y \le$ BIGGER $\Rightarrow$ $0 \le y \le \sqrt{x}$ $\quad +4$

ENDPOINT $\le x \le$ ENDPOINT $\Rightarrow$ $0 \le x \le 4$

2) ANS $= \displaystyle\int_0^4 \int_0^{\sqrt{x}} y\cos(x^2)\,dy\,dx$

$$= \int_0^4 \left[\frac{y^2}{2}\cos(x^2)\right]_{y=0}^{y=\sqrt{x}} dx \quad +4$$

$$= \int_0^4 \frac{x}{2}\cos(x^2)\,dx$$

$$= \left[\frac{\sin(x^2)}{4}\right]_0^4 = \frac{\sin(16)}{4}$$

+2

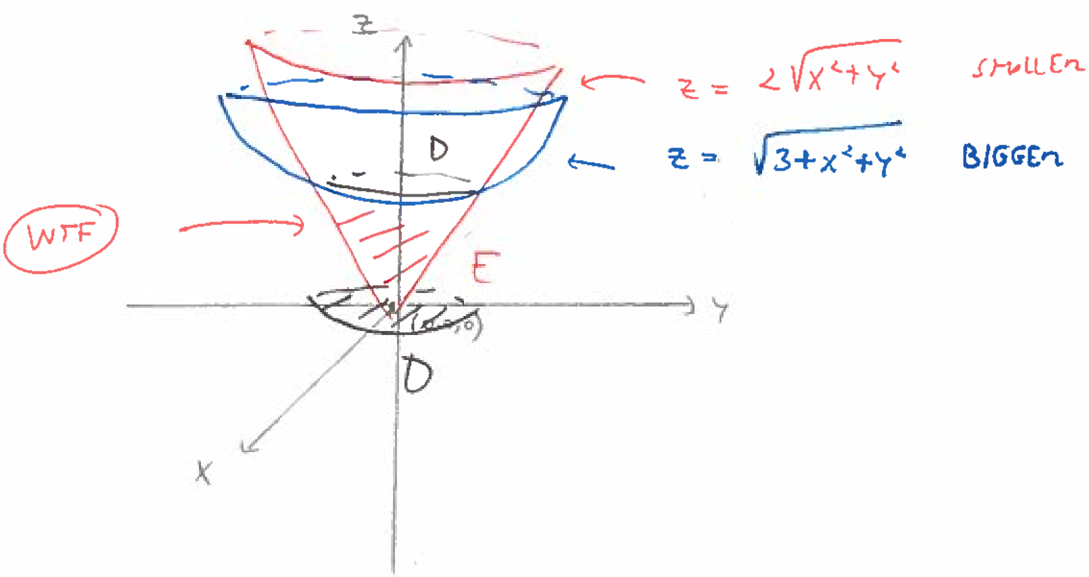
8. *(10 points)* Find the volume of the solid (containing $(0,0,0)$) AND between the following two surfaces

$$z = \sqrt{3 + x^2 + y^2} \text{ and } z = 2\sqrt{x^2 + y^2}$$

[redacted] SIMPLIFY YOUR ANSWER AS MUCH AS POSSIBLE

1) $z = \sqrt{3+x^2+y^2} \Rightarrow z^2 = 3 + x^2+y^2 \Rightarrow -x^2 - y^2 + z^2 = 3$ 2 CUPS

$z = 2\sqrt{x^2+y^2} \Rightarrow z^2 = 4(x^2+y^2) \Rightarrow z^2 = 4x^2+4y^2 \rightsquigarrow$ CONE




2) $V = \iiint_E 1 \, dx\,dy\,dz$ +2

SMALLER $\leq z \leq$ BIGGER

$2\sqrt{x^2+y^2} \leq z \leq \sqrt{3+x^2+y^2}$

SO $V = \iint_D \int_{2\sqrt{x^2+y^2}}^{\sqrt{3+x^2+y^2}} 1 \, dz \, dx\,dz$

$= \iint_D \sqrt{3+x^2+y^2} - 2\sqrt{x^2+y^2} \, dx\,dy$ $+\cancel{4}.2$

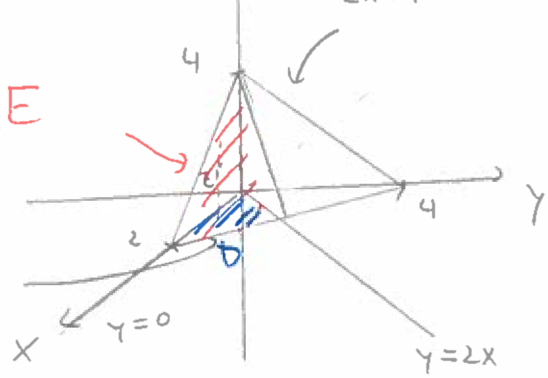
9. *(10 points)* Set up, but do **NOT** evaluate the following integral, where $E$ is the tetrahedron in the first octant bounded by the planes $2x + y + z = 4$, $y = 0$, and $y = 2x$ (Here $dV$ is $dxdydz$ but in any order you prefer)

$$\int\int\int_E xz \, dV$$

1) PICTURE

$2x + y + z = 4 \Rightarrow z = 4 - 2x - y$ BIGGER

$z = 0$ SMALLER




NOTE INTERCEPTS OF $2x + y + z = 4$ ARE

X-INT $(y = 0, z = 0)$ $2x = 4 \Rightarrow x = 2$

Y-INT $(x = 0, z = 0)$ $y = 4$

Z-INT $(x = 0, y = 0)$ $z = 4$

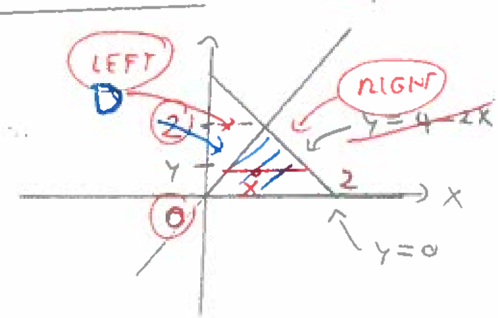
2) SMALLER $\leq z \leq$ BIGGER

$$\boxed{0 \leq z \leq 4 - 2x - y}$$

$x = \frac{y}{2}$

~~$y = 2x$~~

3) FIND D




$\Rightarrow -2x = y - 4$

$\Rightarrow x = -\frac{y}{2} + 2$

NOTE SETTING $z = 0$ IN $2x + y + z = 4$ GIVES $2x + y = 4 \Rightarrow y = 4 - 2x$

WRITE D AS A HORIZONTAL REGION

3) FIND D

INTERSECTION $\sqrt{3+x^2+y^2} = 2\sqrt{x^2+y^2}$

$$3+x^2+y^2 = 4(x^2+y^2)$$

$$3(x^2+y^2) = 3 \qquad +2$$

$$x^2+y^2=1$$

D = DISK OF RADIUS 1

so $$V = \int_0^{2\pi}\int_0^1 \left(\sqrt{3+r^2} - 2\sqrt{r^2}\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\sqrt{3+r^2} - 2r^2 \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{\cancel{2}}{3}\frac{1}{\cancel{2}}(3+r^2)^{\frac{3}{2}} - \frac{2}{3}r^3\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}(4)^{\frac{3}{2}} - \frac{(3)^{\frac{3}{2}}}{3} - \frac{2}{3} + 0 \, d\theta \qquad 4^{\frac{3}{2}} = (\sqrt{4})^3 = 2^3 = 8 \qquad 3^{\frac{3}{2}} = (\sqrt{3})^3 = 3\sqrt{3}$$

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$$= \int_0^{2\pi} \frac{1}{3}8 - \frac{\cancel{3}\sqrt{3}}{\cancel{3}} - \frac{2}{3} \, d\theta \qquad \frac{8}{3} - \frac{2}{3} = \frac{6}{3} = 2$$

$$= \int_0^{2\pi} 2-\sqrt{3} \, d\theta$$

$$= 2\pi(2-\sqrt{3})$$

LEFT $\leq x \leq$ RIGHT

$$\boxed{\frac{y}{2} \leq x \leq -\frac{y}{2}+2}$$

4) FINALLY $2x = 4-2x$ (INTERSECTION)

$\Rightarrow 4x=4$

$\Rightarrow x=1$

$\Rightarrow y=2x=2(1)=2$

SO $\boxed{0 \leq y \leq 2}$

5) PUTTING EVERYTHING TOGETHER, WE GET:

$$\underline{\text{ANS}} = \boxed{\int_0^2 \int_{\frac{y}{2}}^{2-\frac{y}{2}} \int_0^{4-2x-y} xz \, dz\, dx\, dy}$$

(b) *(3 points)* Use (a) with $a = -i$ and the following facts about complex numbers to calculate[1]

$$\int_{-\infty}^{\infty} \cos(x^2)dx \text{ and } \int_{-\infty}^{\infty} \sin(x^2)dx$$

**Fact 1:** $\frac{1}{\sqrt{-i}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$
**Fact 2:** $e^{iz} = \cos(z) + i\sin(z)$ for any $z$
**Fact 3:** If $a + bi = c + di$, then $a = c$ and $b = d$

FROM (a) WITH $a = -i$ WE GET

$$\int_{-\infty}^{\infty} e^{-(-i)x^2} dx = \sqrt{\frac{\pi}{-i}} \quad +1$$

$$\int_{-\infty}^{\infty} e^{ix^2} dx = \frac{\sqrt{\pi}}{\sqrt{-i}}$$

FACT 2 FACT 1

$$\int_{-\infty}^{\infty} \cos(x^2) + i\sin(x^2)\, dx = \sqrt{\pi}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \quad +1$$

$$\int_{-\infty}^{\infty} \cos(x^2)dx + i\int_{-\infty}^{\infty} \sin(x^2)dx = \sqrt{\frac{\pi}{2}} + i\sqrt{\frac{\pi}{2}}$$

FACT 3

$$\int_{-\infty}^{\infty} \cos(x^2)dx = \sqrt{\frac{\pi}{2}} \quad \text{AND} \quad \int_{-\infty}^{\infty} \sin(x^2)dx = \sqrt{\frac{\pi}{2}} \quad +1$$

[1]Technically the result of (a) doesn't apply since $a$ isn't necessarily positive, but surprisingly it gives the correct result!

10. *(10 points)* The Grand Finale!!!

(a) *(7 points)* Using **polar coordinates**, calculate the following integral, where $a > 0$ is a fixed constant

$$\int_{-\infty}^{\infty} e^{-(ax^2)} dx$$

1) LET $\underline{I} = \int_{-\infty}^{\infty} e^{-ax^2} dx$

ALSO $\underline{I} = \int_{-\infty}^{\infty} e^{-ay^2} dy$

2) MULTIPLY $I^2 = \left(\int_{-\infty}^{\infty} e^{-ax^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-ay^2} dy\right)$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-ax^2} dx\right) e^{-ay^2} dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ax^2} e^{-ay^2} dx\,dy \qquad +2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dx\,dy$$

3) $= \int_0^{2\pi}\int_0^{\infty} e^{-ar^2} r\,dr\,d\theta$ +1

(TURN PAGE)

+3 $= \int_0^{2\pi} \left[\frac{e^{-ar^2}}{-2a}\right]_{r=0}^{r=\infty} d\theta = \int_0^{2\pi} \underbrace{\frac{e^{-a(\infty)}}{-2a}}_{=0} + \underbrace{\frac{e^{-a(0)}}{2a}}_{=1} d\theta$

$= \int_0^{2\pi} \frac{1}{2a} d\theta = \frac{1}{\cancel{2}a}(\cancel{2}\pi) = \frac{\pi}{a}$, 4) SO $I^2 = \frac{\pi}{a}$, SO $I = \sqrt{\frac{\pi}{a}}$

+1