SOLUTIONS

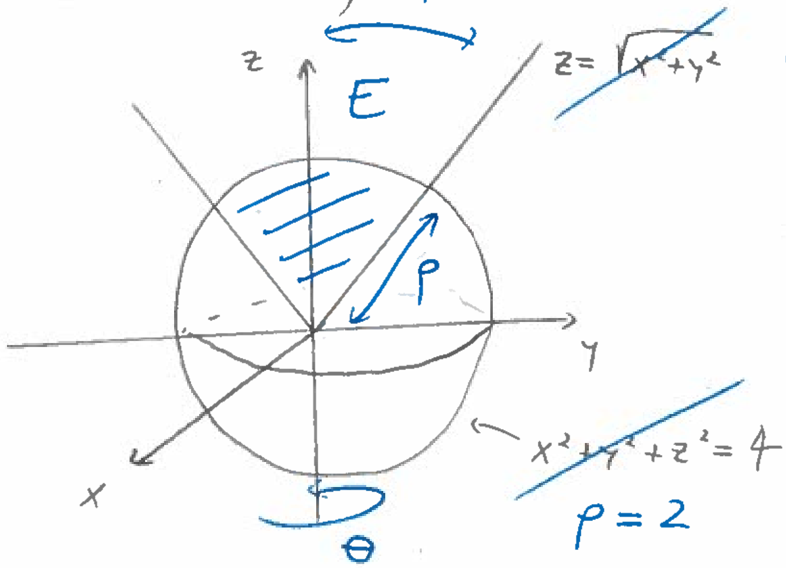
2. *(10 points)* Calculate

$$\iiint_E (x^2+y^2+z^2)^2\, dxdydz$$

Where E is the region inside the surface $\underbrace{x^2+y^2+z^2 \le 4}_{\text{SPHERE}}$ and above the surface $z=\sqrt{x^2+y^2} \to$ CONE

(INCLUDE A PICTURE OF E)

1) PICTURE



2) SPHERICAL COORDS

$$0 \le \rho \le 2$$

$$0 \le \theta \le 2\pi$$

$$0 \le \phi \le \frac{\pi}{4}$$

3) $$\iiint_E (x^2+y^2+z^2)^2\, dxdydz = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{2} (\rho^2)^2 \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$

$$= \left(\int_0^2 \rho^6\, d\rho\right)(2\pi)\left(\int_0^{\pi/4} \sin(\phi)\, d\phi\right)$$

$$= \left[\frac{\rho^7}{7}\right]_0^2 (2\pi) \left[-\cos(\phi)\right]_0^{\pi/4}$$

$$= \frac{2^7}{7}(2\pi)\left(-\frac{\sqrt{2}}{2}+1\right)$$

$$= \boxed{\frac{2^8}{7}(\pi)\left(1-\frac{\sqrt{2}}{2}\right)}$$

$$\left(= \frac{2^7}{7}\pi\,(2-\sqrt{2})\right)$$

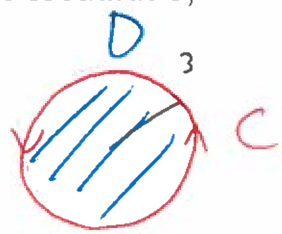
3. *(10 points)* Calculate

$$\int_C F \cdot dr$$

Where $F = \langle \sin(x) + y^2 x, x^2 y + 2x \rangle$

And C is any circle of radius 3, in the counterclockwise direction

1) OPTIONAL :



2) F CONS ?

$$Q_x - P_y = (x^2y + 2x)_x - (\sin(x) + y^2x)_y$$

$$= \cancel{2xy} + 2 - 0 - \cancel{2yx}$$

$$= 2 \neq 0$$

3) BY GREEN ,

$$\int_C F \cdot dr = \iint_D Q_x - P_y \, dx\,dy$$

$$= \iint_D 2 \, dx\,dy$$

$$= 2 \, \text{AREA}(D)$$

$$= 2\pi(3^2) = 18\pi$$

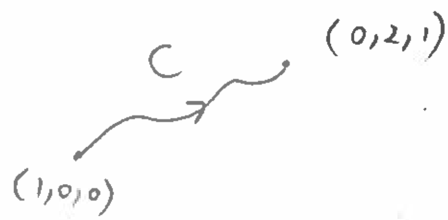
4. *(10 points)* Calculate

$$\int_C F \cdot dr$$

Where $F = \langle y, z\cos(y) + x, \sin(y) \rangle$

And C is any curve from $(1, 1, 0)$ to $(0, 2, 1)$.

1)



2) F CONS ?

$$\text{CURL}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z\cos(y)+x & \sin(y) \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(\sin(y)) - \frac{\partial}{\partial z}(z\cos(y)+x),\ -\frac{\partial}{\partial x}(\sin(y)) + \frac{\partial}{\partial z}(y),\ \frac{\partial}{\partial x}(z\cos(y)+x) - \frac{\partial}{\partial y}(y) \right\rangle$$

$$= \langle \cancel{\cos(y)} - \cancel{\cos(y)},\ \cancel{0}+\cancel{0},\ \cancel{1}-\cancel{1} \rangle$$

$$= \underline{\langle 0, 0, 0 \rangle} \checkmark \qquad \underline{\text{F CONS}}$$

3) FIND f $\qquad F = \nabla f$

$$\Rightarrow \langle Y, Z\cos(Y) + X, \sin(Y) \rangle = \langle f_X, f_Y, f_Z \rangle$$

$$\Rightarrow \begin{cases} f_X = Y \Rightarrow f = \int Y \, dX = XY + \text{JUNK} \\ f_Y = Z\cos(Y) + X \Rightarrow f = \int Z\cos(Y) + X \, dY = Z\sin(Y) + XY + \text{JUNK} \\ f_Z = \sin(Y) \Rightarrow f = \int \sin(Y) \, dZ = Z\sin(Y) + \text{JUNK} \end{cases}$$

$$\Rightarrow \underline{f(X,Y,Z) = XY + Z\sin(Y)}$$

4) BY FTC

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr$$

$$= f(0,2,1) - f(1,1,0)$$

$$= (0)(2) + 1\sin(2) - (1)(1) - 0\sin(1)$$

$$= \boxed{\sin(2) - 1}$$

5. *(10 points)* Find

$$\int\int_S F \cdot d\mathbf{S}$$

Where $F = \langle 1, z, x^2 \rangle$

And S is the surface parametrized by

$$r(u,v) = \langle v, uv, u+v \rangle$$
$$0 \le u \le 2$$
$$0 \le v \le 4$$

With upward orientation (no need to draw S)

1) $r_u = \langle 0, v, 1 \rangle$

$r_v = \langle 1, u, 1 \rangle$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 0 & v & 1 \\ 1 & u & 1 \end{vmatrix} = \langle v-u, 1, \underbrace{-v}_{<0\,!} \rangle$$

USE $\hat{N} = -\langle v-u, 1, -v \rangle = \langle u-v, -1, v \rangle$ INSTEAD!

2) $$\iint_S F \cdot d\vec{S} = \iint_D F \cdot \hat{N} \, du\,dv$$

$$= \int_0^4 \int_0^2 \underbrace{\langle 1, u+v, v^2 \rangle}_{\langle 1, z, x^2 \rangle} \cdot \langle u-v, -1, v \rangle \, du\,dv$$

$$= \int_0^4 \int_0^2 U - V - (U+V) + V^3 \, dU \, dV$$

$$= \int_0^4 \int_0^2 \cancel{U} - V - \cancel{U} - V + V^3 \, dU \, dV$$

$$= (2-0) \int_0^4 -2V + V^3 \, dV$$

$$= 2 \left[-V^2 + \frac{V^4}{4} \right]_0^4$$

$$= 2 \left[-16 + \frac{4^{\cancel{4}\,3}}{\cancel{4}} \right]$$

$$= 2(-16+64)$$

$$= 2(48)$$

$$= 96$$

6. (*10 points*) Use the following change of variables to calculate

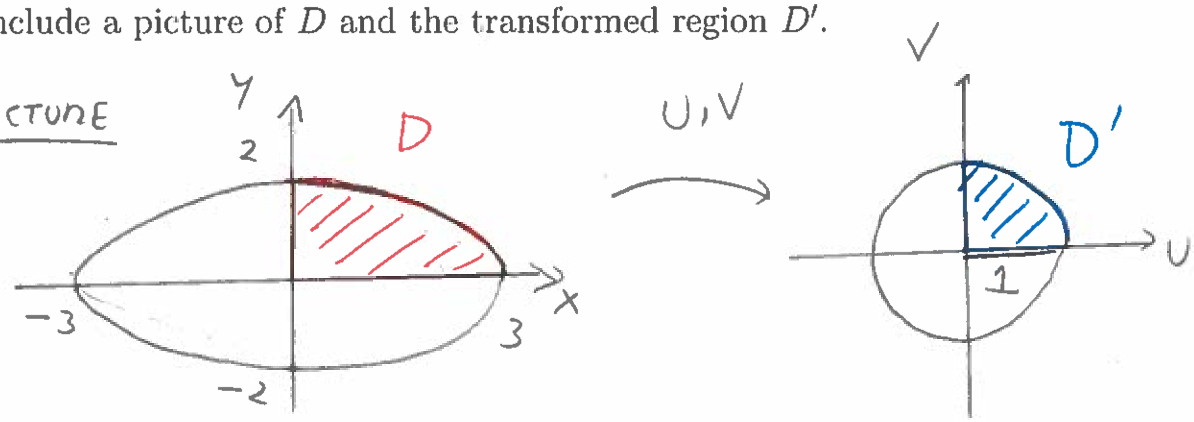
$$\int\int_D (9x^2+4y^2)^{\frac{5}{2}}\, dx dy$$

Where D is the ellipsoid $9x^2+4y^2=1$ in the first quadrant.

$$\begin{cases} u = 3x \\ v = 2y \end{cases}$$

Include a picture of D and the transformed region D'.

1) PICTURE



2) $\begin{cases} U = 3x \\ V = 2Y \end{cases}$

3) FIND D' $9x^2+4Y^2=1 \Rightarrow (3x)^2+(2Y)^2=1$

$\Rightarrow \underline{U^2+V^2=1}$

SO D' IS A QUARTER CIRCLE OF RADIUS 1

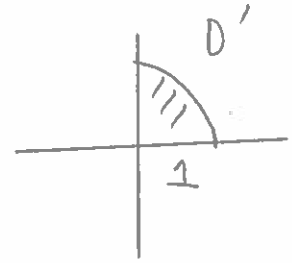
4) $dUdV = \left|\dfrac{dUdV}{dxdY}\right| dxdY = |6|\, dxdY = 6\, dxdY$

$$\frac{dUdV}{dxdY} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial Y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial Y} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6$$

$$dUdV = 6\,dxdy \Rightarrow dxdy = \frac{1}{6}\,dUdV$$

5) $$\iint_D (9x^2+4y^2)^{\frac{5}{2}}\,dxdy.$$

$$= \iint_{D'} (U^2+V^2)^{\frac{5}{2}}\,\frac{1}{6}\,dxdy$$



$$= \int_0^{\pi/2}\int_0^1 (r^2)^{\frac{5}{2}}\,\frac{1}{6}\,r\,dr\,d\theta$$

$$= \frac{1}{6}(\pi/2)\int_0^1 r^5\,r\,dr$$

$$= \frac{\pi}{12}\int_0^1 r^6\,dr$$

$$= \frac{\pi}{12}\left[\frac{r^7}{7}\right]_0^1$$

$$= \frac{\pi}{84}$$

7. *(10 points)* Calculate

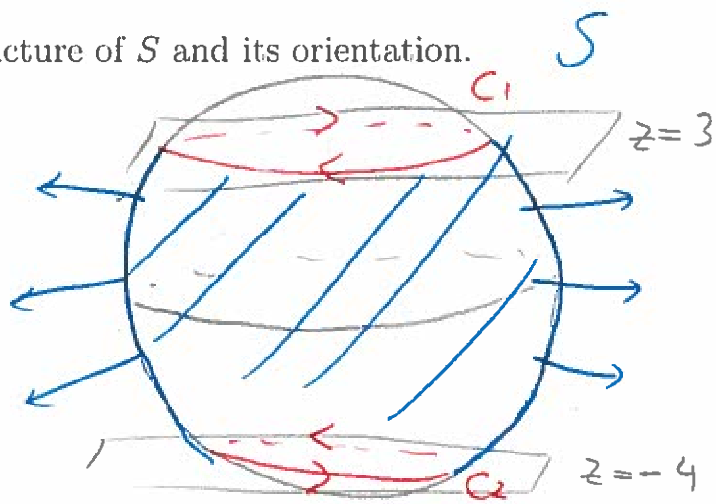
$$\int\int_S \text{curl}(F) \cdot d\mathbf{S}$$

Where $F = \langle x + y, y + e^z, y^5 \rangle$

And S is the portion of the sphere $x^2 + y^2 + z^2 = 25$ *strictly* between the planes $z = 3$ and $z = -4$ (without the top and without the bottom), oriented outwards

Include a picture of S and its orientation.

1) PICTURE



2) BY STOKES, $\iint_S \text{curl}(F) \cdot d\vec{S} = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr$

WHERE C_1 IS CLOCKWISE AND C_2 IS COUNTERCLOCKWISE

3) PARAMETRIZE C_1

(ASSUMING FOR THE MOMENT C_1 IS COUNTERCLOCKWISE)

$x^2+y^2+z^2=25$ AND $z=3 \Rightarrow x^2+y^2+9=25 \Rightarrow x^2+y^2=16$

SO C_1 IS A CIRCLE OF RADIUS 4 IN THE PLANE $z=3$

$r(t) = \langle 4\cos(t), 4\sin(t), 3 \rangle$

CLOCKWISE

$$\int_{C_1} F\cdot dr = -\int_0^{2\pi} F(r(t))\cdot r'(t)\,dt$$

$$= -\int_0^{2\pi} \langle \underbrace{4\cos(t)+4\sin(t),\ 4\sin(t)+e^3,\ 4^5\sin^5(t)}_{\langle x+y,\ y+e^z,\ y^5\rangle} \rangle \cdot \underbrace{\langle -4\sin(t), 4\cos(t), 0\rangle}_{r'(t)}\,dt$$

$$= -\int_0^{2\pi} \cancel{-16\cos(t)\sin(t)} - 16\sin^2(t) + \cancel{16\sin(t)\cos(t)} + 4e^3\cos(t)\,dt$$

$$= \int_0^{2\pi} 16\sin^2(t) - 4e^3\cos(t)\,dt$$

$$= \int_0^{2\pi} 8 - 8\cos(2t) - 4e^3\cos(t)\,dt$$

$$= \left[8t - 4\cancel{\sin(2t)}^{\,0} - 4e^3\cancel{\sin(t)}^{\,0}\right]_0^{2\pi}$$

$$= 8(2\pi) = \underline{16\pi}$$

$x^2+y^2+z^2=25$ & $z=-4 \Rightarrow x^2+y^2+16=25$
$\Rightarrow x^2+y^2=9$

4) PARAMETRIZE C_2

$$r(t) = \langle 3\cos(t), 3\sin(t), -4\rangle$$

$$\int_{C_2} F\cdot dr = \int_0^{2\pi} F(r(t))\cdot r'(t)\,dt$$

$$= \int_0^{2\pi} \langle 3\cos(t)+3\sin(t),\ 3\sin(t)+e^{-4},\ 3^5\sin^5(t)\rangle \cdot \langle -3\sin(t), 3\cos(t), 0\rangle\,dt$$

$$= \int_0^{2\pi} -9\sin^2(t) + 3e^{-4}\cos(t)\,dt$$

$$= \left[-\frac{9}{2}t - \frac{9}{4}\cancel{\sin(2t)}^{\,0} + 3e^{-4}\cancel{\sin(t)}^{\,0}\right]_0^{2\pi}$$

$$= \left(-\frac{9}{2}\right)(2\pi) = \underline{-9\pi}$$

5) $$\iint_S F\cdot d\vec{S} = 16\pi - 9\pi = 7\pi$$

8· ~~3~~. *(10 points)* Calculate

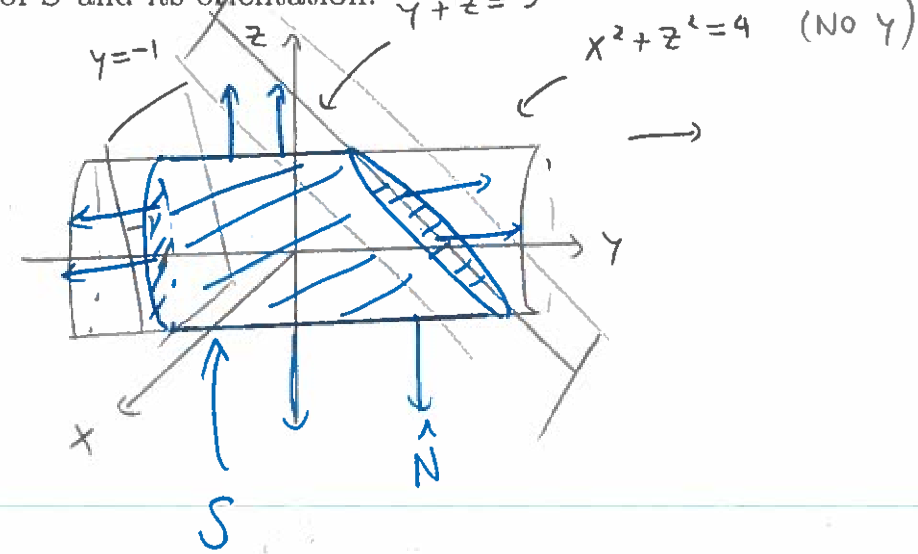
$$\iint_S F \cdot d\mathbf{S}$$

Where $F = \langle x + \sin(y), y + \sin(z), z + \sin(x) \rangle$

And S is the boundary of the region enclosed by $x^2 + z^2 = 4$, $y = -1$, and $y + z = 3$, oriented outwards (including the top/bottom/sides)

Include a picture of S and its orientation.

1) PICTURE



2) DIVERGENCE THEOREM

$$\iint_S F \cdot d\vec{S} = \iiint_E \mathrm{DIV}(F)\, dx\,dy\,dz$$

$$\mathrm{DIV}(F) = (X + \sin(Y))_x + (Y + \sin(Z))_Y + (Z + \sin(X))_Z$$

$$= 1 + 1 + 1$$

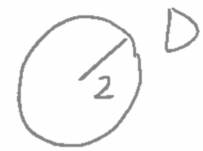
$$= 3$$

$$= \iiint_E 3 \; dx\,dy\,dz$$

$\cancel{y+z=3} \Rightarrow y = 3 - z$

INEQUALITIES

$$\begin{cases} -1 \le y \le \overset{3 - r\sin(\theta)}{\cancel{3-z}} \\ 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{cases}$$



(From $x^2+y^2=4$)

$$= \int_0^{2\pi}\int_0^{2}\int_{-1}^{3-r\sin(\theta)} 3 \; r\,dy\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} 3r\left(3 - r\sin(\theta) + 1\right) dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} 3r\left(4 - r\sin(\theta)\right) dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} 12r - \left(3r^2\sin(\theta)\right) dr\,d\theta$$

$$= \int_0^{2\pi} \left[6r^2 - r^3\sin(\theta) \right]_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} 24 - 8\sin(\theta)\,d\theta$$

$$= \left[24\theta + \cancel{8\cos(\theta)}^{\,0}\right]_0^{2\pi} = 24(2\pi) = 48\pi$$

9. (*10 points*) Calculate

$$\int_C F \cdot dr$$

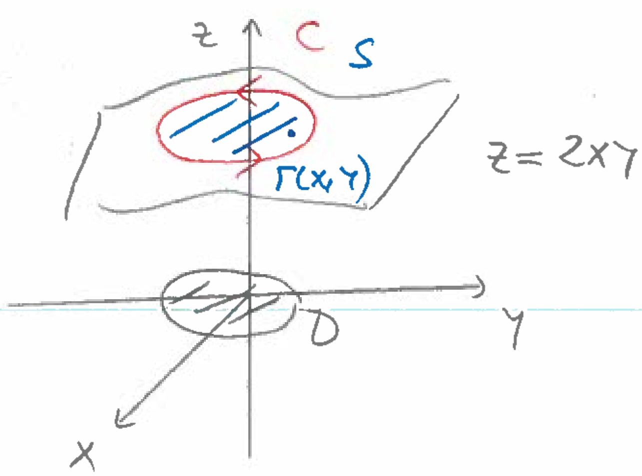
Where $F = \langle 1, x + yz, xy - \sqrt{z} \rangle$

And C is the curve parametrized by

$$r(t) = \langle 2\cos(t), 2\sin(t), 8\cos(t)\sin(t) \rangle \quad 0 \leq t \leq 2\pi$$

Hint: C lies on the surface $z = 2xy$

1) PICTURE (OPTIONAL)



2) BY STOKES' : $\int_C F \cdot dr = \iint_S \text{curl}(F) \cdot d\vec{S}$

3) $\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & x+yz & xy-\sqrt{z} \end{vmatrix}$

$= \left\langle \frac{\partial}{\partial y}(xy-\sqrt{z}) - \frac{\partial}{\partial z}(x+yz), -\frac{\partial}{\partial x}(xy-\sqrt{z}) + \frac{\partial}{\partial z}(1), \frac{\partial}{\partial x}(x+yz) - \frac{\partial}{\partial y}(1) \right\rangle$

$= \underline{\langle x-y, -y, 1\rangle}$

4) <u>PARAMETRIZE S</u> (S = INSIDE OF C)

$r(x,y) = \langle x, y, 2xy\rangle$

$$\left.\begin{aligned} r_x &= \langle 1, 0, 2y\rangle \\ r_y &= \langle 0, 1, 2x\rangle \end{aligned}\right\} \quad r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 2y \\ 0 & 1 & 2x \end{vmatrix}$$

$$= \langle -2y, -2x, \underbrace{1}_{\geq 0}\rangle$$

5) $\displaystyle\int_C F\cdot dr = \iint_S \text{curl}(F)\cdot d\vec{S}$

$$= \iint_D \langle x-y, -y, 1\rangle \cdot \langle -2y, -2x, 1\rangle \, dx\,dy$$

D = DISK OF RADIUS 2

$$= \iint_D -\cancel{2xy} + 2y^2 + \cancel{2xy} + 1 \quad dx\,dy$$

$$= \int_0^{2\pi}\int_0^2 (2r^2 \sin^2(\theta) + 1)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 2r^3 \sin^2(\theta) + r \, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^4}{2}\sin^2(\theta) + \frac{r^2}{2}\right]_0^2 dr\,d\theta = \int_0^{2\pi} 8\sin^2(\theta) + 2 \, d\theta$$

$$= \int_0^{2\pi} 4 - 4\cos(2\theta) + 2 \, d\theta = \left[6\theta - 2\cancel{\sin(2\theta)}\right]_0^{2\pi} = 12\pi$$

10. *(10 points, 5 points each)* The two parts are independent of each other

(a) Suppose $f(x,y)$ satisfies $f_{xx} + f_{yy} = 0$

Let C be any circle of radius 2, oriented counterclockwise.

Calculate:

$$\int_C (f_y)\,dx - (f_x)\,dy$$

$(P = f_y,\ Q = -f_x)$

GREEN

$$= \iint_D Q_x - P_y \, dx\,dy$$

$$= \iint_D (-f_x)_x - (f_y)_y \, dx\,dy$$

$$= \iint_D - f_{xx} - f_{yy} \, dx\,dy$$

$$= \iint_D - \underbrace{(f_{xx} + f_{yy})}_{0} \, dx\,dy$$

$$= 0$$

(b) Let S be any closed surface, oriented outwards, and let E be the inside of S.

$\langle 0,0,0 \rangle$

Suppose $f(x, y, z, t)$ satisfies $f_{tt} = \Delta f$ in E and $\nabla f =$ [redacted] on S.

Let $M(t) = \iiint_E f(x, y, z, t)\,dxdydz$

Show [redacted] $M_{tt}(t) = 0$

Note: Here $\Delta f = f_{xx} + f_{yy} + f_{zz}$ and $\nabla f = \langle f_x, f_y, f_z \rangle$, and $M''(t)$ is the second derivative of M with respect to t.

$$M_{tt}(t) = \left(\iiint_E f(x,y,z,t)\,dxdydz \right)_{tt}$$

$$= \iiint_E \left(f(x,y,z,t)\right)_{tt}\,dxdydz$$

$f_{tt} = \Delta f$

$$= \iiint_E f_{tt}\,dxdydz$$

$$= \iiint_E \Delta f\,dxdydz$$

DIV THM

$$= \iiint_E \mathrm{DIV}(\nabla f)\,dxdydz$$

$$= \iint_S \underbrace{\nabla f}_{0} \cdot d\vec{S}$$

$$= 0$$