SOLUTIONS

2. *(10 points)* Calculate

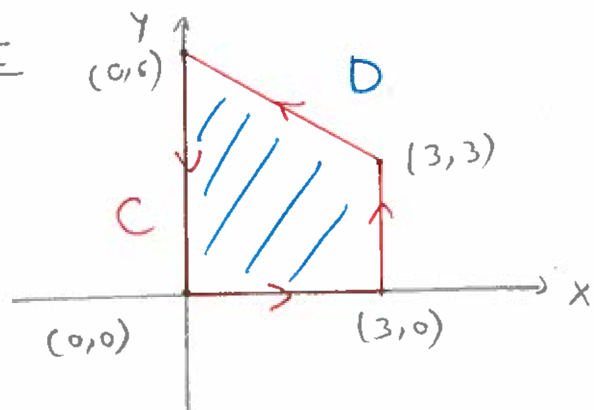
$$\int_C F \cdot dr$$

Where $F = \langle xy, y^3 \rangle$

And C is the polygon connecting the points $(0,0)$, $(3,0)$, $(3,3)$, $(0,6)$, oriented counterclockwise.

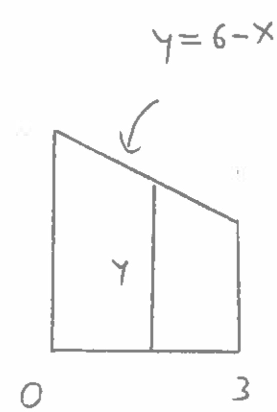
Include a picture of C and its orientation.

1) PICTURE



2) F CONS ? $Q_x - P_y = (y^3)_x - (xy)_y = -x \neq 0$

3) BY GREEN ,



$$\int_C F \cdot dr = \iint_D Q_x - P_y \, dx\,dy$$

$$= \int_0^3 \int_0^{6-x} -x \, dy\,dx$$

$$= \int_0^3 -x(6-x)\,dx$$

$$= \int_0^3 -6x + x^2 \, dx$$

$$= \left[-3x^2 + \frac{x^3}{3} \right]_0^3$$

$$= -3(3^2) + \frac{3^3}{3}$$

$$= -27 + 9$$

$$= -18$$

3. *(10 points)* Calculate

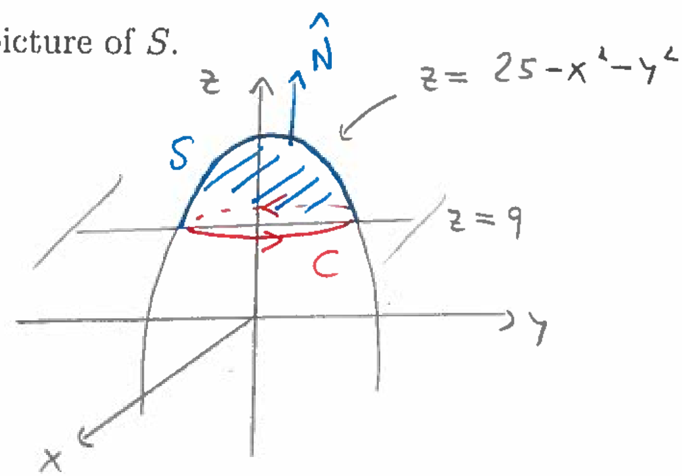
$$\int\int_S \operatorname{curl}(F)\cdot d\mathbf{S}$$

Where $F = \langle x+z, x+y, x^3\rangle$

And S is the portion of the surface $z = 25 - x^2 - y^2$ strictly above the plane $z = 9$ (so without the bottom), oriented upwards.

Include a picture of S.

1) PICTURE



2) BY STOKES $\iint_S \operatorname{curl}(F)\cdot d\vec{S} = \int_C F\cdot dr$

3) PARAMETRIZE C $z = 25 - x^2 - y^2$ AND $z = 9$

$\Rightarrow 25 - x^2 - y^2 = 9 \Rightarrow \underline{x^2 + y^2 = 16}$

SO C IS A CIRCLE OF RADIUS 4 IN THE PLANE $z = 9$

ORIENTED COUNTERCLOCKWISE

$\Rightarrow r(t) = \langle 4\cos(t), 4\sin(t), 9\rangle \quad (0 \le t \le 2\pi)$

4) $$\iint_S \operatorname{curl}(F)\cdot d\vec{S} = \int_C F\cdot dr$$

$$= \int_0^{2\pi} F(r(t))\cdot r'(t)\,dt$$

$$= \int_0^{2\pi} \underbrace{\langle 4\cos(t)+9,\ 4\cos(t)+4\sin(t),\ 64\cos^3(t)\rangle}_{\langle x+2,\ x+y,\ x^3\rangle}\cdot\underbrace{\langle -4\sin(t),\ 4\cos(t),\ 0\rangle}_{r'(t)}\,dt$$

$$= \int_0^{2\pi} -16\cancel{\cos(t)\sin(t)} - 36\sin(t) + 16\cos^2(t) + \cancel{16\cos(t)\sin(t)} + 0\,dt$$

$$= \int_0^{2\pi} -36\sin(t) + 16\cos^2(t)\,dt$$

$$= \left[36\cancel{\cos(t)}\right]_0^{2\pi} + \int_0^{2\pi} 8 + 8\cos(2t)\,dt$$

$$= 8(2\pi) + \left[4\cancel{\sin(2t)}\right]_0^{2\pi}$$

$$= 16\pi$$

4. *(10 points)* Calculate

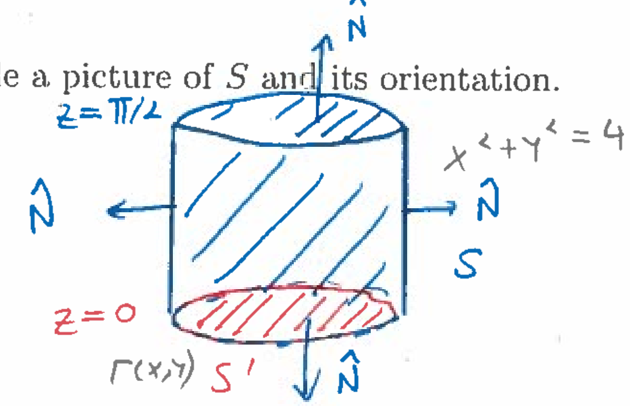
$$\iint_S F \cdot d\mathbf{S}$$

Where $F = \langle x, y + z^3, \cos(z) \rangle$ WITH TOP, BUT

S is the portion of the surface $x^2 + y^2 = 4$ inside the region $0 < z \leq \frac{\pi}{2}$ oriented outwards (so without the bottom).

Include a picture of S and its orientation.

1) PICTURE



2) SINCE S IS NOT CLOSED, LET S' = BOTTOM OF CYLINDER

3) THEN $S + S'$ IS CLOSED, SO BY THE DIVERGENCE THEOREM

$$\iint_{S+S'} F \cdot d\vec{S} = \iiint_E \mathrm{DIV}(F)\, dx\,dy\,dz$$

$$\mathrm{DIV}(F) = (x)_x + (y + z^3)_y + (\cos(z))_z = 1 + 1 - \sin(z) = 2 - \sin(z)$$

$$= \iiint_E 2 - \sin(z)\, dx\,dy\,dz$$

$$= \int_0^{2\pi} \int_0^{2} \int_0^{\pi/2} (2 - \sin(z))\, r \, dz\, dr\, d\theta$$

$$= 2\pi \left(\int_0^2 r\,dr \right) \left(\int_0^{\pi/2} 2 - \sin(z)\,dz \right)$$

$$= 2\pi \left[\frac{r^2}{2} \right]_0^2 \left[2z + \cos(z) \right]_0^{\pi/2} = (2\pi)(2)\left(2\left(\frac{\pi}{2}\right) + 0 - 0 - 1 \right)$$

$$= 4\pi(\pi - 1) = \underline{4\pi^2 - 4\pi}$$

4) $$\iint_{S+S'} F \cdot d\vec{s} = \iint_S F \cdot d\vec{s} + \iint_{S'} F \cdot d\vec{s}$$

$$\Rightarrow \iint_S F \cdot d\vec{s} = \iint_{S+S'} F \cdot d\vec{s} - \iint_{S'} F \cdot d\vec{s} = 4\pi^2 - 4\pi - \iint_{S'} F \cdot d\vec{s}$$

5) $\iint_{S'} F \cdot d\vec{s}$: <u>PARAMETRIZE S'</u>

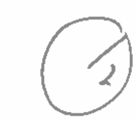
$$r(x,y) = \langle x, y, 0 \rangle \Rightarrow r_x = \langle 1,0,0 \rangle, \; r_y = \langle 0,1,0 \rangle,$$

$$\hat{N} = r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \cancel{\langle 0,0,1 \rangle} \quad \langle 0,0,-1 \rangle$$

($\hat{N}$ NEEDS TO POINT <u>DOWNWARDS</u>)

$$\iint_{S'} F \cdot d\vec{s} = \iint_D \underbrace{\langle x, y+0^3, \cos(0) \rangle}_{\langle x, y+z^3, \cos(z) \rangle} \cdot \underbrace{\langle 0,0,-1 \rangle}_{\hat{N}} \,dx\,dy$$

$$= \iint_D -1 \,dx\,dy \qquad D = \text{DISK OF RAD } 2:$$



$$= \int_0^{2\pi}\int_0^2 -1 \; r\,dr\,d\theta$$

$$= -\pi(2^2) = -4\pi$$

6) <u>CONCLUSION</u> $$\iint_S F \cdot d\vec{s} = \underbrace{\left(4\pi^2 - 4\pi\right)}_{S+S'} - \underbrace{(-4\pi)}_{S'} = 4\pi^2$$

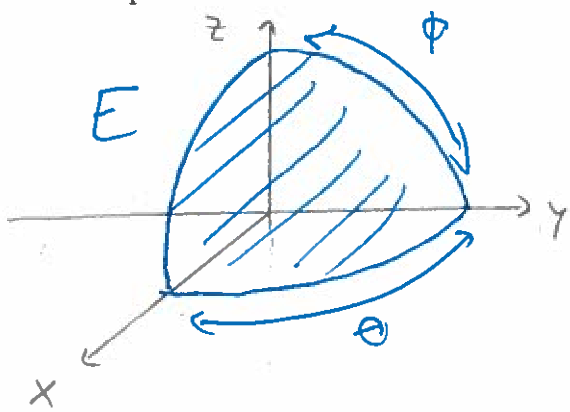
5. *(10 points)* Evaluate

$$\int\int\int_E e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dxdydz$$

Where E is the region inside the surface $x^2+y^2+z^2 \le 4$ in the first octant.

Include a picture of E.

1) PICTURE



2) SPHERICAL COORDINATES

$0 \le \rho \le 2$

$0 \le \theta \le \pi/2$

$0 \le \phi \le \pi/2$

3) $\displaystyle\iiint_E e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dx\,dy\,dz = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} e^{(\rho^2)^{\frac{3}{2}}}\rho^2 \sin(\phi)\,d\rho\,d\theta\,d\phi$

$\displaystyle = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} e^{\rho^3}\rho^2 \sin(\phi)\,d\rho\,d\theta\,d\phi$

$\displaystyle = \left(\int_0^2 e^{\rho^3}\rho^2\,d\rho\right)\left(\frac{\pi}{2}\right)\left(\int_0^{\pi/2}\sin(\phi)\,d\phi\right)$

$$= \left[\frac{1}{3} e^{\rho^3}\right]_0^2 \cdot \left(\frac{\pi}{2}\right) \left[-\cos(\phi)\right]_0^{\pi/2}$$

$$= \left(\frac{1}{3} e^8 - \frac{1}{3}\right) \frac{\pi}{2} (1)$$

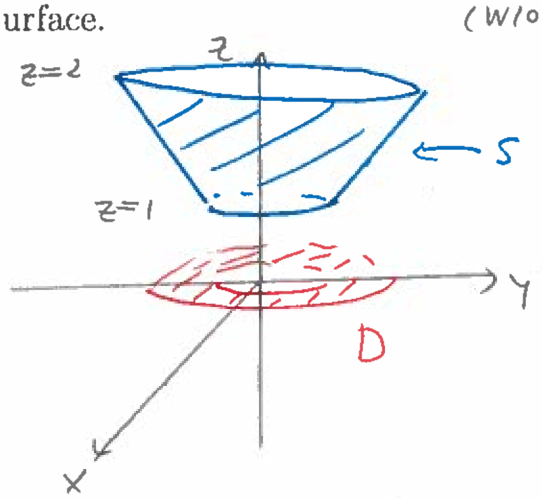
$$= \boxed{\frac{\pi}{6} \left(e^8 - 1\right)}$$

$\iint_S z^2\, dS$, WHERE S IS THE

6. *(10 points)* Find ~~the area of the~~ portion of the surface $z^2 = x^2 + y^2$ between $z = 1$ and $z = 2$. Include a picture of the surface. (W/O TOP OR BOTTOM)

CONE

1)



2) PARAMETRIZE S $\quad r(x,y) = \langle x, y, \sqrt{x^2+y^2} \rangle$

3) $r_x = \langle 1, 0, \dfrac{2x}{2\sqrt{x^2+y^2}} \rangle = \langle 1, 0, \dfrac{x}{\sqrt{x^2+y^2}} \rangle$

$r_y = \langle 0, 1, \dfrac{2y}{2\sqrt{x^2+y^2}} \rangle = \langle 0, 1, \dfrac{y}{\sqrt{x^2+y^2}} \rangle$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & \dfrac{x}{\sqrt{x^2+y^2}} \\ 0 & 1 & \dfrac{y}{\sqrt{x^2+y^2}} \end{vmatrix} = \langle -\frac{x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \rangle$$

$dS = \| r_x \times r_y \| \, dx\,dy$

$$= \sqrt{\left(\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right) + 1}\; dx\,dy = \sqrt{\frac{x^2+y^2}{x^2+y^2} + 1}\; dx\,dy$$

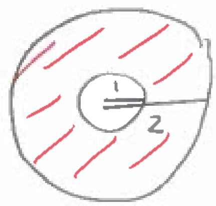
$$= \sqrt{1+1}\; dx\,dy = \sqrt{2}\; dx\,dy$$

4) $$\iint_S z^2\,dS = \iint_D \left(\sqrt{x^2+y^2}\right)^2 \sqrt{2}\,dx\,dy$$

$$= \iint_D \sqrt{2}\,(x^2+y^2)\,dx\,dy$$

⚠ D = SHADOW UNDER S

D =



(ANNULUS WITH RADIUS 1 & 2,
SINCE $z=1 \Rightarrow \sqrt{x^2+y^2}=1 \Rightarrow r=1$
$z=2 \Rightarrow \sqrt{x^2+y^2}=2 \Rightarrow r=2$)

$$= \int_0^{2\pi}\int_1^2 \sqrt{2}\, r^2\, r\,dr\,d\theta$$

$$= \sqrt{2}\,(2\pi)\left(\int_1^2 r^3\,dr\right)$$

$$= 2\pi\sqrt{2}\left[\frac{r^4}{4}\right]_1^2$$

$$= 2\pi\sqrt{2}\left[\frac{16}{4}-\frac{1}{4}\right]$$

$$= 2\pi\sqrt{2}\left(\frac{15}{4}\right)$$

$$= \frac{15\pi}{2}\sqrt{2}$$

7. *(10 points)* Calculate

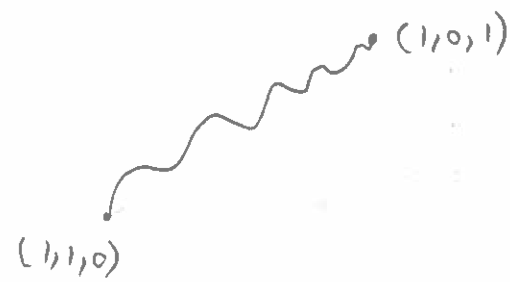
$$\int_C \mathbf{F} \cdot dr$$

Where $F = \langle 2xz + y^2, 2xy, x^2 + 3z^2 \rangle$

ANY CURVE

And C is [redacted] connecting the points $(1,1,0)$ and $(1,0,1)$.

1) PICTURE



2) F CONS. ?

$$\mathrm{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz + y^2 & 2xy & x^2 + 3z^2 \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(x^2+3z^2) - \frac{\partial}{\partial z}(2xy),\ -\frac{\partial}{\partial x}(x^2+3z^2) + \frac{\partial}{\partial z}(2xz+y^2),\ \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(2xz+y^2) \right\rangle$$

$$= \langle \cancel{0} - \cancel{0}, -\cancel{2x} + \cancel{2x}, \cancel{2y} - \cancel{2y} \rangle$$

$$= \langle 0, 0, 0 \rangle \checkmark$$

3) <u>FIND f</u> $\quad F = \nabla f \Rightarrow \langle 2xz + y^2, 2xy, x^2 + 3z^2 \rangle$

$= \langle f_x, f_y, f_z \rangle$

$\Rightarrow f_x = 2xz + y^2 \Rightarrow f = \int 2xz + y^2 \, dx = x^2 z + xy^2 + \text{JUNK}$

$f_y = 2xy \Rightarrow f = \int 2xy \, dy = xy^2 + \text{JUNK}$

$f_z = x^2 + 3z^2 \Rightarrow f = \int x^2 + 3z^2 \, dz = x^2 z + z^3 + \text{JUNK}$

$\underline{f(x,y,z) = xy^2 + x^2 z + z^3}$

4) BY FTC :

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(1,0,1) - f(1,1,0)$$

$$= (1)(0^2) + (1^2)(1) + 1^3$$

$$- (1)(1^2) - (1^2)(0) - 0^3$$

$$= 2 - 1$$

$$= 1$$

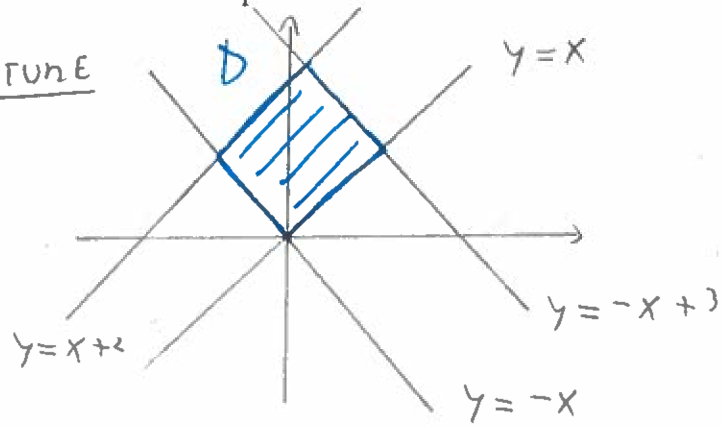
8. *(10 points)* Calculate

$$\int\int_D (y+x)\, e^{(y-x)(y+x)}\, dx\,dy$$

Where D is the region between the lines $y = x$, $y = x + 2$, $y = -x$ and $y = -x + 3$.

Include a picture of D.

1) PICTURE



2) $U = y + x$

$V = y - x$

3) NOTICE $y = x \Rightarrow y - x = 0 \Rightarrow V = 0$

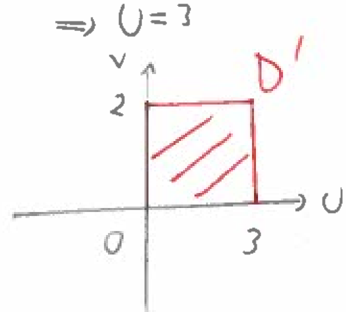
$y = x + 2 \Rightarrow y - x = 2 \Rightarrow V = 2$

$y = -x \Rightarrow y + x = 0 \Rightarrow U = 0$

$y = -x + 3 \Rightarrow y + x = 3 \Rightarrow U = 3$

SO D' BECOME $0 \leq U \leq 3$

$0 \leq V \leq 2$



4) JACOBIAN

$$dUdV = \left|\frac{dUdV}{dx\,dy}\right| dx\,dy = |2|\,dx\,dy = 2\,dx\,dy$$

$$\frac{dUdV}{dx\,dy} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

so $dUdV = 2\,dx\,dy \Rightarrow dx\,dy = \frac{1}{2}\,dUdV$

5) $$\iint_D (y+x)\,e^{y^2-x^2}\,dx\,dy = \iint_D (y+x)\,e^{(y-x)(y+x)}\,dx\,dy$$

$$= \iint_{D'} U e^{VU}\left(\tfrac{1}{2}\,dUdV\right)$$

$$= \frac{1}{2}\int_0^2\int_0^3 U e^{VU}\,dUdV = \frac{1}{2}\int_0^3\int_0^2 U e^{VU}\,dV\,dU$$

$$= \frac{1}{2}\int_0^3 \left[\frac{\cancel{U}e^{VU}}{\cancel{U}}\right]_{V=0}^{V=2} dU$$

$$= \frac{1}{2}\int_0^3 e^{2U} - e^{0U}\,dU$$

$$= \frac{1}{2}\int_0^3 e^{2U} - 1\,dU$$

$$= \frac{1}{2}\left[\frac{e^{2U}}{2} - U\right]_0^3$$

$$= \frac{1}{2}\left(\frac{e^6}{2} - 3 - \frac{1}{2}\right) = \boxed{\frac{e^6}{4} - \frac{7}{4}} \quad \boxed{\frac{1}{4}(e^6 - 7)}$$

9. *(10 points)* Calculate

$$\int_C F \cdot dr$$

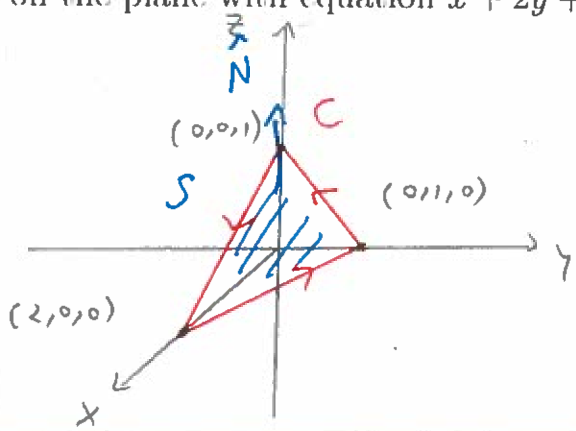
Where $F = \left\langle e^{x^2}, x + y^2 + z, y + \sin(z) \right\rangle$

And C is the triangle connecting the points $(2,0,0)$, $(0,1,0)$, $(0,0,1)$, in the counterclockwise direction.

Include a picture of C

Note: C lies on the plane with equation $x + 2y + 2z = 2$.

1) PICTURE



2) BY STOKES, $\int_C F \cdot dr = \iint_S \text{curl}(F) \cdot d\vec{S}$

3) $\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{x^2} & x+y^2+z & y+\sin(z) \end{vmatrix}$

$= \left\langle \frac{\partial}{\partial y}(y+\sin(z)) - \frac{\partial}{\partial z}(x+y^2+z),\ -\frac{\partial}{\partial x}(y+\sin(z)) + \frac{\partial}{\partial z}(e^{x^2}),\ \frac{\partial}{\partial x}(x+y^2+z) - \frac{\partial}{\partial y}(e^{x^2}) \right\rangle = \underline{\langle 0,0,1 \rangle}$

4) PARAMETRIZE S SINCE S LIES IN THE PLANE $x+2y+2z=2$.

$$\Rightarrow \quad 2z = 2 - x - 2y$$

$$\Rightarrow \quad z = 1 - \frac{x}{2} - y$$

$$r(x,y) = \langle x, y, 1-\frac{x}{2} - y \rangle$$

$$\left.\begin{array}{l} r_x = \langle 1, 0, -\frac{1}{2} \rangle \\ r_y = \langle 0, 1, -1 \rangle \end{array}\right\} \qquad r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -1/2 \\ 0 & 1 & -1 \end{vmatrix}$$

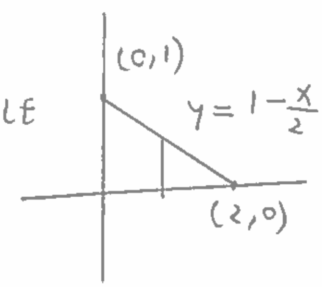
$$\Rightarrow \quad \hat{N} = \langle \frac{1}{2}, 1, \underbrace{1}_{\geq 0} \rangle$$

5) $$\int_C F \cdot dr = \iint_S \text{curl}(F) \cdot d\vec{S}$$

$$= \iint_D \underbrace{\langle 0, 0, 1 \rangle}_{\text{curl}(F)} \cdot \underbrace{\langle \frac{1}{2}, 1, 1 \rangle}_{\hat{N}} \, dx\,dy$$

$$= \iint_D 1 \, dx\,dy$$

D: TRIANGLE



$$= \int_0^2 \int_0^{1-\frac{x}{2}} dy\,dx$$

$$= \int_0^2 1 - \frac{x}{2} \, dx$$

$$= \left[x - \frac{x^2}{4} \right]_0^2$$

$$= 2 - \frac{2^2}{4} = 2 - 1 = 1$$

10. *(10 points)* Use the adult version of the surface integral (the one that involves **n**) to calculate

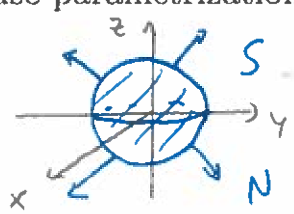
$$\int\int_S F \cdot d\mathbf{S}$$

Where $F = \left\langle \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle$

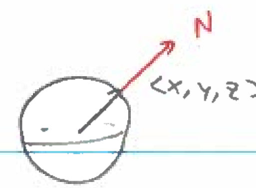
And S is the sphere centered at $(0,0,0)$ and radius r, oriented outwards.

Do not use parametrizations

1) PICTURE



2) $\iint_S F \cdot d\vec{S} = \iint_S F \cdot N \, dS$



FOR THE SPHERE OF RADIUS r, $N = \frac{\langle x, y, z \rangle}{r}$

$$= \iint_S \underbrace{\left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle}_{F} \cdot \underbrace{\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle}_{N} dS$$

$$= \iint_S \frac{1}{r^4} \underbrace{(x^2+y^2+z^2)}_{r^2} \, dS$$

$$= \frac{1}{r^2} \iint_S 1 \, dS = \frac{1}{r^2} \text{AREA}(S) = \frac{1}{\cancel{r^2}} 4\pi \cancel{r^2} = 4\pi$$