HOMEWORK 1 – AP SOLUTIONS

AP 1: Let P_n be the proposition $\int_0^\infty x^n e^{-x} dx = n!$

Base Case: $n = 0$

$$\begin{aligned}\int_0^\infty x^0 e^{-x} dx &= \int_0^\infty e^{-x} dx \\ &= \left[-e^{-x}\right]_0^\infty \\ &= -e^{-\infty} + e^0 \\ &= 1 \\ &= 0! \checkmark\end{aligned}$$

Inductive Step: Suppose P_n is true, that is $\int_0^\infty x^n e^{-x} dx = n!$.

Show P_{n+1} is true, that is $\int_0^\infty x^{n+1} e^{-x} dx = (n+1)!$

$$\begin{aligned}&\int_0^\infty x^{n+1} e^{-x} dx \\ \overset{IBP}{=} &\left[-x^{n+1} e^{-x}\right]_0^\infty - \int_0^\infty (n+1) x^n \left(-e^{-x}\right) dx \\ = &\left(\lim_{x\to\infty} -x^{n+1} e^{-x}\right) + 0^{n+1} e^{-0} + (n+1) \int_0^\infty x^n e^{-x} dx \\ = &(n+1) n! \qquad \text{By the hint with } k = n+1 \text{ and the } \textbf{inductive hypothesis} \\ = &(n+1)! \checkmark\end{aligned}$$

Date: Friday, September 3, 2021.

Hence P_{n+1} is true, and therefore P_n is true for all n, that is $\int_0^\infty x^n e^{-x} dx = n!$

AP 2: Let P_n be the proposition:

$$\sin(x) + \sin(3x) + \cdots + \sin((2n-1)x) = \frac{1-\cos(2nx)}{2\sin(x)}$$

Base Case: $n = 1$. Since $2(1) - 1 = 1$, the left-hand-side just becomes $\sin(x)$, whereas the right-hand-side is:

$$\begin{aligned}\frac{1-\cos(2x)}{2\sin(x)} &= \frac{1-\cos^2(x)+\sin^2(x)}{2\sin(x)} \\ &= \frac{\sin^2(x)+\sin^2(x)}{2\sin(x)} \\ &= \frac{2\sin^2(x)}{2\sin(x)} \\ &= \sin(x) \checkmark\end{aligned}$$

Inductive Step: Suppose P_n is true, that is

$$\sin(x) + \sin(3x) + \cdots + \sin((2n-1)x) = \frac{1-\cos(2nx)}{2\sin(x)}$$

Show P_{n+1} is true, that is:

$$\sin(x) + \sin(3x) + \cdots + \sin((2(n+1)-1)x) = \frac{1-\cos(2(n+1)x)}{2\sin(x)}$$

$$
\begin{aligned}
&\sin(x) + \sin(3x) + \cdots + \sin((2(n+1) - 1)x) \\
=&\sin(x) + \cdots + \sin((2n-1)x) + \sin((2n+1)x) \\
=&\frac{1-\cos(2nx)}{2\sin(x)} + \sin((2n+1)x) \qquad \textbf{(By the inductive hypothesis)} \\
=&\frac{1-\cos(2nx) + \sin((2n+1)x)2\sin(x)}{2\sin(x)} \\
=&\frac{1-\cos(2nx) + \sin(2nx+x)2\sin(x)}{2\sin(x)} \\
=&\frac{1-\cos(2nx) + 2\sin(x)\left(\sin(2nx)\cos(x) + \cos(2nx)\sin(x)\right)}{2\sin(x)} \\
=&\frac{1-\cos(2nx) + \sin(2nx)2\cos(x)\sin(x) + 2\cos(2nx)\sin^2(x)}{2\sin(x)} \\
=&\frac{1-\cos(2nx)(1-2\sin^2(x)) + \sin(2nx)\sin(2x)}{2\sin(x)} \\
=&\frac{1-\cos(2nx)(\cos^2(x)+\sin^2(x)-2\sin^2(x)) + \sin(2nx)\sin(2x)}{2\sin(x)} \\
=&\frac{1-\cos(2nx)(\cos^2(x)-\sin^2(x)) + \sin(2nx)\sin(2x)}{2\sin(x)} \\
=&\frac{1-\cos(2nx)\cos(2x) + \sin(2nx)\sin(2x)}{2\sin(x)} \\
=&\frac{1-\cos(2nx+2x)}{2\sin(x)} \\
=&\frac{1-\cos((2n+1)x)}{2\sin(x)}
\end{aligned}
$$

Therefore P_{n+1} is true, and hence P_n is true for all n, that is

$$\sin(x) + \sin(3x) + \cdots + \sin((2n-1)x) = \frac{1-\cos(2nx)}{2\sin(x)}$$

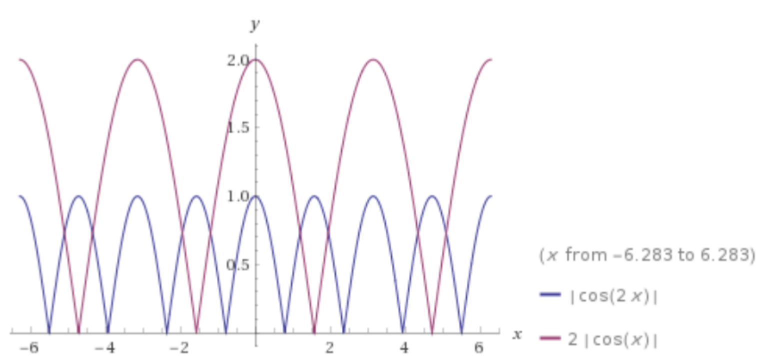
AP 3: The statement is FALSE

Let $n = 2$, then we'd have to prove or disprove whether

$$|\cos(2x)| \leq 2\,|\cos(x)|$$

But now let $x = \frac{\pi}{2}$, then the left-hand-side becomes $|\cos(\pi)| = |-1| = 1$ whereas the right-hand-side is $2\left|\cos\left(\frac{\pi}{2}\right)\right| = 0$.

In fact, if you graph both functions, you can see that one is not above the other one (courtesy wolframalpha):



Note: If you try to imitate the proof in the book, you would eventually get $|\cos((n+1)x)| \leq |\cos(nx)|\,|\cos(x)| + |\sin(nx)|\,|\sin(x)|$, but all you can deduce from that is $|\cos((n+1)x)| \leq |\cos(nx)| + |\sin(x)|$, which doesn't combine as nicely as the proof with sin.

AP 4

$$\begin{aligned}
0 =& \emptyset \\
1 =& 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{\emptyset\} \\
2 =& 1 \cup \{1\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} \\
3 =& \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}$$