Pei yan Li, uni: pl2801.

MATH 4062 – HOMEWORK 1

Note: The problems refer to the Rudin textbook. See below for hints

- **Chapter 7:** 2, 7

I proved it

Note: For 2 you're allowed to use problem 1 (without proof)

Please **also** do the additional problems below.

Additional Problem 1: Consider the following sequence of functions f_n on $[0,1]$, sometimes called the **growing steeple**

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ 2 - nx & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \leq x \leq 1 \end{cases}$$

Show that f_n converges pointwise to 0, but not uniformly to 0.

Additional Problem 2: Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of uniformly continuous functions and $f_n \to f$ uniformly as $n \to \infty$. Show that f is uniformly continuous.

Date: Due: Friday, July 8, 2022.

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

Proof of 1: Not required in Homework, but I proved it anyway.

uniformly convergent sequence:

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ for all x, $|f_n(x) - f(x)| < \varepsilon$ when $n \geq N$

then $f_n(x) - \varepsilon \leq f(x) \leq f_n(x) + \varepsilon$ for all x

we know that f_n is a bounded function,

then f is also a bounded function

denote $\|f\| < C$ I use this in the proof of question 2

Then $\exists\ \varepsilon = 1$ s.t. $\exists N_0 \in \mathbb{N}$ for all x $|f_n(x) - f(x)| < 1$ when $n \geq N_0$

then for all $n > N_0$, $\|f_n\| < C + 1$

We know that $f_1, f_2, \cdots f_{N_0}$ is bounded,

denote $\|f_i\| \leq M_i$, $i = 1, 2, \cdots, N_0$

then take $M = \max\{M_1, M_2, \cdots M_{N_0}, C+1\}$

M is a uniform bound of $\{f_n\}$, thus $\{f_n\}$ is uniformly bounded.

Proof of 2 is on next page

2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

let f_n uniformly coverage to f and g_n uniformly coverage to g

① then $\forall \varepsilon > 0$ $\exists N_f \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$, $n \geq N_f$, $\forall x \in E$

$\exists N_g \in \mathbb{N}$ s.t $|g_n(x) - g(x)| < \frac{\varepsilon}{2}$, $n \geq N_g$, $\forall x \in E$

$\exists N = \max\{N_f, N_g\} \in \mathbb{N}$

s.t. $|f_n(x) + g_n(x) - (f(x) + g(x))|$

$\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \leq \varepsilon$ $n \geq N, \forall x \in E$

which means $f_n + g_n$ converges uniformly on E (to $f+g$)

② by 1, we know that $\{f_n\}$ & $\{g_n\}$ are uniformly bounded

From the proof of 1, we also know that f and g are bounded as well

take $M > 0$ s.t. $|f| < M$, $|g| < M$ and $|f_n| < M$, $|g_n| < M$ for all n

$\forall \varepsilon > 0$ $\exists N_1 \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2M}$ for all x and $n > N_1$

$\exists N_2 \in \mathbb{N}$ s.t. $|g_n(x) - g(x)| \leq \frac{\varepsilon}{2M}$ for all x and $n > N_2$

$\exists N = \max(N_1, N_2)$ s.t.

$$|f_n g_n(x) - fg(x)| = |f_n(x) g_n(x) - f_n(x) g(x) + f_n(x) g(x) - f(x) g(x)|$$

$$\leq |f_n(x)| |g_n(x) - g(x)| + |g(x)| |f_n(x) - f(x)|$$

$$\leq M |g_n(x) - g(x)| + M |f_n(x) - f(x)|$$

$$\leq \varepsilon \quad \text{for all } n > N \text{ and all } x$$

which means $f_n g_n$ converges uniformly on E. (to fg)

7. For $n = 1, 2, 3, \ldots$, x real, put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n \to \infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

① Fixed x, $\lim_{n\to\infty} f_n(x) = 0$, so $f_n(x)$ converges pointwise to $f \equiv 0$

② In this part, we prove $\{f_n\}$ converges uniformly (yellow highlight part)

$\forall \varepsilon > 0$ $\exists N > \frac{1}{4\varepsilon^2}$ s.t. $\forall n \geq N$, $\frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{N}} < \frac{1}{2\sqrt{\frac{1}{4\varepsilon^2}}} = \varepsilon$

Thus, we have

$$|f_n(x) - 0| = \left|\frac{x}{1+nx^2}\right| = \left|\frac{1}{\frac{1}{x}+nx}\right| \leq \frac{1}{2\sqrt{n}} < \varepsilon,$$

$\forall n \geq N$, $\forall x \in \mathbb{R}$

note: $\left|\frac{1}{x} + nx\right| = \frac{1}{|x|} + n|x|$

$$= \frac{1}{(\sqrt{|x|})^2} + \left(\sqrt{n|x|}\right)^2 \geq 2\sqrt{n}$$

note $a^2+b^2 \geq 2ab$

thus $\left|\frac{1}{\frac{1}{x}+nx}\right| \leq \frac{1}{2\sqrt{n}}$

③ 1) $f(x) \equiv 0$, $f'(x) \equiv 0$

2) $f_n'(x) = \frac{(1-nx^2) - 2nx^2}{(1-nx^2)^2} = \frac{1-3nx^2}{(1-nx^2)^2}$

when $x = 0$, $\lim_{n\to\infty} f_n'(x) = 1$

$x \neq 0$, $\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1-3nx^2}{n^2x^4 - 2nx^2 + 1}$

$$= \lim_{n\to\infty} \frac{\frac{1}{n} - 3x^2}{nx^4 - 2x^2 + \frac{1}{n}} = 0$$

Thus, combine 1) and 2)

we conclude that $f'(x) = \lim_{n\to\infty} f_n'(x)$ when $x \neq 0$

but not equal when $x = 0$

Additional Problem 1: Consider the following sequence of functions f_n on $[0,1]$, sometimes called the **growing steeple**

$$f_n(x) = \begin{cases} nx & \text{if } 0 \le x \le \frac{1}{n} \\ 2 - nx & \text{if } \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \le x \le 1 \end{cases}$$

Show that f_n converges pointwise to 0, but not uniformly to 0.

① Converges Pointwise :

$\forall$ fixed x , $\forall \varepsilon > 0$ by Archimedean Properity, $\exists n \in \mathbb{N}$ s.t. $\frac{2}{n} < x$

So by this n, $\frac{2}{n} \le x \le 1$, $f_n(x) = 0$, then,

$|f_n(x) - 0| = |0 - 0| = 0 < \varepsilon$

which means it converges pointwise to 0 (by highlight part)

② not uniformly

$\exists$ $\varepsilon = \frac{1}{2}$, for $\forall N > 0$ $N \in \mathbb{N}$,

$\exists$ $x = \frac{1}{N}$ s.t.

$f_N(\frac{1}{N}) = 1$ where $|f_N(x) - 0| = 1 > \varepsilon = \frac{1}{2}$

which means it is not uniformly converges (by highlight part)

Additional Problem 2: Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of uniformly continuous functions and $f_n \to f$ uniformly as $n \to \infty$. Show that f is uniformly continuous.

$\forall \varepsilon > 0$ $\exists n \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all x

(Because $f_n \to f$ uniformly as $n \to \infty$)

For same n and ε

$\exists \delta > 0$ s.t. $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$ for any $|x - y| < \delta$

(Because fixed n, f_n is uniformly continuous)

Then:

$$|f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{when } |x - y| < \delta$$

By the yellow high light part, we could conclude that

f is uniformly continuous.