HOMEWORK 10 – AP SOLUTIONS

AP 1

(a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\left(\frac{\sin(h)}{h}\right) \\
&= \sin(x) \times 0 + \cos(x) \times 1 \\
&= \cos(x)
\end{aligned}$$

(b)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \cos(x)\left(\frac{\cos(h) - 1}{h}\right) - \sin(x)\left(\frac{\sin(h)}{h}\right) \\
&= \cos(x) \times 0 - \sin(x) \times 1 \\
&= -\sin(x)
\end{aligned}$$

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(c)

$$\begin{aligned}f'(x) &= \left(\frac{\sin(x)}{\cos(x)}\right)'\\ &= \frac{\cos(x)\cos(x) - \sin(x)\left(-\sin(x)\right)}{\cos^2(x)}\\ &= \frac{cos^2(x) + \sin^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)}\\ &= \sec^2(x)\end{aligned}$$

(d)

$$\left(\tan^{-1}(y)\right)' = \frac{1}{(\tan)'(x)} = \frac{1}{\sec^2(x)} = \frac{1}{1+\tan^2(x)} = \frac{1}{1+\tan^2(\tan^{-1}(y))} = \frac{1}{1+y^2}$$

Hence $\left(\tan^{-1}(x)\right)' = \frac{1}{1+x^2}$

AP 2

Differentiating both sides of the equation, we get:

$$\left(\ln\left(\frac{f(x)}{g(x)}\right)\right)' = (\ln(f(x)))' - (\ln(g(x)))'$$

$$\frac{\left(\frac{f(x)}{g(x)}\right)'}{\left(\frac{f(x)}{g(x)}\right)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \left(\frac{f(x)}{g(x)}\right)\left(\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right)$$

$$= \left(\frac{f(x)}{g(x)}\right)\left(\frac{f'(x)g(x) - g'(x)f(x)}{f(x)g(x)}\right)$$

$$= \frac{f'(x)g(x)f(x) - g'(x)f(x)f(x)}{f(x)g(x)g(x)}$$

$$= \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

AP 3

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x}$$

However, we have

$$\frac{1}{x} = x\left(\frac{1}{x}\right)^2 = 2x\frac{1}{2}\left(\frac{1}{x}\right)^2 \leq 2x\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{x}\right)^n = 2xe^{\frac{1}{x}}$$

The middle step follows because $\frac{1}{2!}\left(\frac{1}{x}\right)^2$ is one term of the series, and the last step follows from the definition of e^x as a series (from a previous homework). Therefore we have:

$$0 \leq \frac{1}{x} \leq 2xe^{\frac{1}{x}} \Rightarrow 0 \leq \frac{e^{-\frac{1}{x}}}{x} \leq 2x$$

And by the squeeze theorem we get $\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x} = 0$

Therefore $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = 0$ and the limit as $x \to 0^-$ simply follows because $f(x) = 0$ for negative x. Hence $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$.

For part (b), since $f^{(n)}(0) = 0$ for all n, the Maclaurin series of f becomes $\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0$, so it's the zero function, which is not a good approximation to f at all!

AP 4

(a) Let $g(x) = \frac{f(x+a)}{f(x)}$, then

$$g'(x) = \frac{f'(x+a)f(x) - f(x+a)f'(x)}{(f(x))^2} = \frac{f(x+a)f(x) - f(x+a)f(x)}{(f(x))^2} = 0$$

So $g'(x) = 0$, so $g(x) = C$, but notice that:

$$C = g(0) = \frac{f(a)}{f(0)} = f(a)$$

Hence $C = f(a)$, and we get that $g(x) = f(a)$, so

$$\frac{f(x+a)}{f(x)} = f(a)$$

And multiplying by $f(x)$, we get:

$$f(x+a) = f(x)f(a)$$

(b) Let $g(x) = f(-x)f(x)$, then

$$g'(x) = -f'(-x)f(x) + f(-x)f'(x) = -f(-x)f(x) + f(-x)f(x) = 0$$

Hence $g(x) = C$.

$$\text{But } C = g(0) = f(0)f(0) = 1 \times 1 = 1$$

So $g(x) = 1$, and $f(-x)f(x) = 1$, and:

$$f(-x) = \frac{1}{f(x)}$$

(c) Let $g(x) = \frac{f(ax)}{f(x)^a}$, then:

$$\begin{aligned} g'(x) =& \frac{f'(ax)a(f(x))^a - f(ax)af(x)^{a-1}f'(x)}{f(x)^{2a}} \\ =& \frac{af(ax)f(x)^a - af(ax)f(x)^{a-1}f(x)}{f(x)^{2a}} \\ =& \frac{af(ax)f(x)^a - af(ax)f(x)^a}{f(x)^{2a}} \\ =& 0 \end{aligned}$$

So $g(x) = C$, but:

$$C = g(0) = \frac{f(0)}{f(0)^a} = \frac{1}{1} = 1$$

So $g(x) = 1$, so $\frac{f(ax)}{f(x)^a} = 1$, so:

$$f(ax) = f(x)^a$$

AP 5

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} &\overset{\hat{H}}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{2\sqrt{x^2+1}}\right)2x}{1} \\
&= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \\
&\overset{\hat{H}}{=} \lim_{x\to\infty} \frac{1}{\left(\frac{1}{2\sqrt{x^2+1}}\right)2x} \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}
\end{aligned}$$

Oh no, we end up getting back to where we started from!

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} &= \lim_{x\to\infty} \frac{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x} \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}{x} \quad \sqrt{x^2} = |x| = x \text{ Since } x > 0 \\
&= \lim_{x\to\infty} \frac{x\sqrt{1+\frac{1}{x^2}}}{x} \\
&= \lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}} \\
&= \sqrt{1+0} \\
&= 1
\end{aligned}$$