HOMEWORK 10 – SELECTED BOOK SOLUTIONS

28.6

Let $\epsilon > 0$ be given, let $\delta = \epsilon$, then if $|x| < \delta = \epsilon$, then

$$\left| x \sin\left(\frac{1}{x}\right) - 0 \right| = \left| x \sin\left(\frac{1}{x}\right) \right| \leq |x| < \epsilon \checkmark$$

Therefore f is continuous at $x = 0$. However:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right) \qquad \text{DNE}$$

Hence f is not differentiable at $x = 0$.

28.15

First let's show the result in the hint:

Date: Friday, November 19, 2021.

$$\begin{aligned}\binom{n}{k-1}+\binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!}+\frac{n!}{k!(n-k)!}\\ &= \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{n-k+1}+\frac{1}{k}\right)\\ &= \frac{n!}{(k-1)!(n-k)!}\left(\frac{k+(n-k+1)}{k(n-k+1)}\right)\\ &= \frac{n!}{(k-1)!(n-k)!}\left(\frac{n+1}{k(n-k+1)}\right)\\ &= \frac{n!(n+1)}{(k-1)!k(n-k)!(n+1-k)!}\\ &= \frac{(n+1)!}{k!(n+1-k)!}\\ &= \binom{n+1}{k}\end{aligned}$$

Now let P_n be the proposition

$$(fg)^{(n)}(a)=\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(a)g^{(n-k)}(a)$$

Base case: $n=0$, then the left hand side is just $fg(a)$ and the right-hand-side is

$$\sum_{k=0}^{0}\binom{n}{k}f^{(k)}(a)g^{(0-k)}(a)=\binom{0}{0}f^{(0)}(a)g^{(0-0)}(a)=f(a)g(a)\checkmark$$

Now suppose P_n is true and show P_{n+1} is true, but:

$$
\begin{aligned}
(fg)^{(n+1)}(a) &= \left((fg)^{(n)}\right)'(a) \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k)}(a)g^{(n-k)}(a)\right)' \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)}(a)g^{(n-k)}(a) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)g^{(n-k+1)}(a) \quad \text{(Prod. Rule)} \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)}(a)g^{(n-(k-1))}(a) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)g^{(n-k+1)}(a) \\
&= \sum_{k=1}^{n} \binom{n}{k-1} f^{(k)}(a)g^{n-(k-1)}(a) + \binom{n}{n} f^{(n+1)}(a)g(a) \\
&+ \binom{n}{0} f^{(0)}(a)g^{(n+1)}(a) + \sum_{k=1}^{n} \binom{n}{k} f^{(k)}(a)g^{(n-k+1)}(a) \\
&= f(a)g^{(n+1)}(a) + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k}\right) f^{(k)}(a)g^{n+1-k}(a) + f^{(n+1)}(a)g(a) \\
&= \binom{n+1}{0} f(a)g^{(n+1)}(a) + \sum_{k=1}^{n} \binom{n+1}{k} f^{(k)}(a)g^{n+1-k}(a) \\
&+ \binom{n+1}{n+1} f^{(n+1)}(a)g(a) \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(a)g^{(n+1-k)}(a) \checkmark
\end{aligned}
$$

Hence P_{n+1} is true and so P_n is true for all n $\square$

29.4

Let $h(x) = f(x)e^{g(x)}$, then

$$h(a) = f(a)e^{g(a)} = 0, \qquad h(b) = f(b)e^(g(b)) = 0$$

Since $h(a) = h(b)$, by Rolle's Theorem, there is some c in (a,b) such that $h'(c) = 0$. However:

$$h'(x) = \left(f(x)e^{g(x)}\right)' = f'(x)e^{g(x)} + f(x)e^{g(x)}g'(x) = e^{g(x)}\left(f'(x) + f(x)g'(x)\right)$$

$$\begin{aligned} h'(c) &= 0 \\ e^{g(c)}\left(f'(c) + f(c)g'(c)\right) &= 0 \\ f'(c) + f(c)g'(c) &= 0 \qquad (\text{Since } e^{g(c)} > 0) \end{aligned}$$

Which is what we wanted □

29.5

Given x and h, use the above assumption with $x + h$ instead of y to conclude that

$$|f(x) - f(x+h)| \leq (x - (x+h))^2 = h^2$$

Dividing by h, this implies that

$$\left|\frac{f(x+h) - f(x)}{h}\right| \leq h$$

Since $\lim_{h \to 0} h = 0$, by the squeeze theorem, this implies that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0$$

Hence $f'(x) = 0$ for all x and so f is constant □

29.18(a)

STEP 1: Let's first show by induction on n that

$$|s_{n+1} - s_n| \leq a^n |s_1 - s_0|$$

The base case $n = 0$ gives $|s_1 - s_0| \leq a^0 |s_1 - s_0| \checkmark$.

For the inductive step, assume $|s_{n+1} - s_n| \leq a^n |s_1 - s_0|$, then:

$$\begin{aligned}
|s_{n+2} - s_n| &= |f(s_{n+1}) - f(s_n)| \\
&= \left| \frac{f(s_{n+1}) - f(s_n)}{s_{n+1} - s_n} \right| |s_{n+1} - s_n| \\
&= |f'(c)| \, |s_{n+1} - s_n| \qquad \text{(Mean Value Theorem)} \\
&\leq a \, |s_{n+1} - s_n| \qquad \text{(Definition of } a\text{)} \\
&\leq a a^n |s_1 - s_0| \qquad \text{(Inductive Hypothesis)} \\
&= a^{n+1} |s_1 - s_0| \checkmark
\end{aligned}$$

STEP 2: Then WLOG, if $n > m$, it follows that

$$\begin{aligned}
|s_n - s_m| &= |s_n - s_{n-1} + s_{n_1} - s_{n-2} + \cdots + s_{m+1} - s_m| \\
&\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \cdots + |s_{m+1} - s_m| \\
&\leq a^{n-1} |s_1 - s_0| + a^{n-2} |s_1 - s_0| + \cdots + a^m |s_1 - s_0| \\
&= \left(a^{n-1} + a^{n-2} + \cdots + a^m\right) |s_1 - s_0| \\
&= a^m \left(a^{n-m-1} + a^{n-m-2} + \cdots + 1\right) |s_1 - s_0| \\
&\leq a^m \left|1 + a^2 + a^3 + \cdots\right| |s_1 - s_0| \qquad (a > 0) \\
&= a^m \frac{1}{1-a} |s_1 - s_0| \qquad \text{(Geometric Series)} \\
&= a^m \frac{|s_1 - s_0|}{1-a}
\end{aligned}$$

STEP 3: Now if $\epsilon > 0$ is given since $\lim_{m\to} a^m = 0$ (since $a < 1$) there is N such that if $m > N$, then

$$a^m < \epsilon \left(\frac{1-a}{|s_1 - s_0|} \right)$$

(Note $s_1 \neq s_0$, unless $f(s_0) = s_0$, in which case we have a fixed point) With that N, if $m, n > N$, then WLOG $n > m$, and then

$$|s_n - s_m| \leq a^m \frac{|s_1 - s_0|}{1-a} < \epsilon \left(\frac{1-a}{|s_1 - s_0|} \right) \frac{|s_1 - s_0|}{1-a} = \epsilon$$

Therefore (s_n) is Cauchy and since $\mathbb{R}$ is complete, (s_n) converges to some s □

29.18(b)

By definition of s_n we have $s_n = f(s_{n-1})$

Now $s_n \to s$ and hence $s_{n-1} \to s$, but since f is continuous, this implies $f(s_{n-1}) \to f(s)$

So taking the limit as $n \to \infty$ in the equation $s_n = f(s_{n-1})$ we ultimately get $s = f(s)$, and therefore f has a fixed point □

30.6

Applying the hint and L'Hôpital's rule (since $e^x \to \infty$), we get

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{f(x)e^x}{e^x}\\ &\stackrel{\hat{H}}{=} \lim_{x\to\infty} \frac{(f(x)e^x)'}{(e^x)'}\\ &= \lim_{x\to\infty} \frac{f'(x)e^x + f(x)e^x}{e^x}\\ &= \lim_{x\to\infty} f'(x) + f(x)\\ &=L\end{aligned}$$

Therefore $f(x) \to L$, and then

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} (f'(x) + f(x)) - f(x) = \lim_{x\to\infty} f'(x) + f(x) - \lim_{x\to\infty} f(x) = L - L = 0 \checkmark$$

30.7(a)

$$f(x) = x + \cos(x)\sin(x) \geq x - 1$$

Since $\lim_{x\to\infty} x - 1 = \infty$, by comparison of limits, $\lim_{x\to\infty} f(x) = \infty$

$$g(x) = e^{\sin(x)}(x + \cos(x)\sin(x)) \geq e^{-1}(x-1)$$

Since $\lim_{x\to\infty} e^{-1}(x-1) = \infty$, by comp. of limits, $\lim_{x\to\infty} g(x) = \infty$

30.7(b)

$$\begin{aligned}f'(x) &=1 - \sin(x)\sin(x) + \cos(x)\cos(x)\\ &=1 - \sin^2(x) + \cos^(x)\\ &= \cos^2(x) + \cos^2(x)\\ &=2\cos^2(x)\end{aligned}$$

$$\begin{aligned}
g'(x) =& e^{\sin(x)}\cos(x)\left(x+\cos(x)\sin(x)\right)+e^{\sin(x)}\left(x+\cos(x)\sin(x)\right)' \\
=& e^{\sin(x)}\cos(x)\left(x+\cos(x)\sin(x)\right)+e^{\sin(x)}\left(2\cos^2(x)\right) \\
=& e^{\sin(x)}\cos(x)\left(x+\cos(x)\sin(x)+2\cos(x)\right) \\
=& e^{\sin(x)}\cos(x)\left(2\cos(x)+f(x)\right)
\end{aligned}$$

30.7(c)

$$\begin{aligned}
\frac{f'(x)}{g'(x)} =& \frac{2\cos^2(x)}{e^{\sin(x)}\cos(x)\left(2\cos(x)+f(x)\right)} \\
=& \frac{2\cos(x)}{e^{\sin(x)}\left(2\cos(x)+f(x)\right)} \\
=& \frac{2e^{-\sin(x)}\cos(x)}{2\cos(x)+f(x)}
\end{aligned}$$

30.7(d)

The first part follows from the squeeze theorem, since the numerator is bounded by $2e$ whereas the denominator $2\cos(x)+f(x)\to\infty$ (by comparison of limits). As for the second limit:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f(x)}{e^{\sin(x)}f(x)} = \lim_{x\to\infty}\frac{1}{e^{\sin(x)}} \quad \text{Does not Exist}$$